§1 Recall: Intervals in an interval are used to compute areas under curves.

We define them using Riemann sums:

\[
\text{Area under the curve and the } x\text{-axis } = \int_a^b f(x) \, dx
\]

(Here: \( f(x) \geq 0 \) so signed area is usual area)

We approximate this area by covering it with rectangles.

**Step 1:** Break the interval \([a,b]\) into \(n\) intervals of equal length.
(We will make \( \Delta x \) very small!) This is called a **regular partition**.

\[
\Delta x = \frac{b-a}{n}
\]

\( x_k \) are grid points \( k = 1, \ldots, n \)

\( x_k = a + k \Delta x \).

**Step 2:** Pick a point \( x_k^* \) in \([x_{k-1}, x_k]\) for every \( k = 1, \ldots, n \).

For every \( k \), we form the rectangle with base \([x_{k-1}, x_k]\)

\[ 	ext{a height } = f(x_k^*) \text{ call it net}_k \]

\[ \text{Area (net}_k \text{) } = f(x_k^*) \Delta x. \]

**Proof:** The Riemann sum associated with this data is

\[ \text{sum of areas } = \sum_{k=1}^{n} f(x_k^*) \Delta x + \sum_{k=1}^{n} f(x_k^*) \Delta x + \ldots + \sum_{k=1}^{n} f(x_k^*) \Delta x. \]

\[ = \sum_{k=1}^{n} f(x_k^*) \Delta x \]

We can pick the points \( x_k^* \) arbitrarily or following some rules:

1. **Pick left pt of every interval:** \( x_k^* = x_{k-1} \) \( \Rightarrow \) **left riemann sum**.
2. **Right pt** \( x_k^* = x_k \) \( \Rightarrow \) **right**.
3. **Midpoint** \( x_k^* = \frac{x_{k-1} + x_k}{2} \) \( \Rightarrow \) **midpoint**.

If \( x_k^* \) in \([x_{k-1}, x_k]\) is arbitrary, then we call the sum a **general riemann sum**.

**Definition:**

\[ \int_a^b f(x) \, dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k \]

\( \Delta x = \max \{ \Delta(x_k^*) : k = 1, \ldots, n \} \)

We say \( f \) is integrable whenever the (RHS) limit exists.

**Theorem:** If \( f \) continuous on \([a,b]\), then \( f \) is integrable on \([a,b]\).
§2 Double integrals: Fix \( R = [a, b] \times [c, d] \) as \( \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\} \) rectangle in \( \mathbb{R}^2 \) and \( f : f : R \to \mathbb{R} \) when \( f(x, y) \geq 0 \).

**Goal:** Find the Volume of the solid bounded by the graph of \( f \) and \( R \) (if \( f(x, y) \) has arbitrary sign, we get a signed volume).

- We want to mimic what we did for functions of one variable, so we need 2 steps:
  1. **Step 1:** Break the rectangle \( R \) into \( N \) rectangular subregions with sides parallel to the \( x \)-axis & \( y \)-axis, respectively. The lengths of the \( k^{th} \) rectangle are \( \Delta x_k \) & \( \Delta y_k \), respectively. 
    
\[
\text{Area (rect}_k\text{)} = \Delta x_k \Delta y_k := \Delta A_k 
\]

   - The rectangles form a partition of \( R \)
   - The grid need not be regular (\( \Delta x_k \) can be different for different \( k \)'s & the same for \( \Delta y_k \)'s)
   - We can order the rectangles (e.g., from left to right & bottom to top)

  2. **Step 2:** On the \( k^{th} \) rectangle, we pick any point \( (x_k^*, y_k^*) \)

We build the \( k^{th} \) box with
- base = \( k^{th} \) rectangle
- height = \( f(x_k^*, y_k^*) \)

\[
\text{Volume (k^{th} box)} = f(x_k^*, y_k^*) \Delta A_k 
\]

\[
\Sigma \left\{ \text{Volumes of all boxes} \right\} = \sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta x_k \Delta y_k 
\]

- This sum approximates the volume of the solid. The smaller the (diagonal of all rectangles), the better the approximation (so, as \( N \to \infty \) we should \( \Delta = \text{diag} \to 0 \) set small!

- Write \( \Delta = \max(\text{diag}) = \max \left\{ \sqrt{\Delta x_k^2 + \Delta y_k^2} \right\} \)

\[ \int \int_R f(x, y) \, dA := \lim_{\Delta \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*) \Delta A_k \quad \text{(double integral of } f \text{ over } R) \]

\( f \) is integrable on \( R \) if the limit exists for all partitions of \( R \) and all choices of \( (x_k^*, y_k^*) \) within those partitions.

**Note:** If \( f(x, y) \geq 0 \), the integral is the volume bounded by \( R \) and the graph of \( f \).
§3 Iterated integrals:

Q: How to compute \( \iint_R f(x,y) \, dA \)?

A: Use slicing method

- Slice along \( yz \)-planes
  \[ A(x) = \int_c^d f(x,y) \, dy \]
  \[ A(y) = \int_a^b f(x,y) \, dx \]

- Cross sectional area = \( A(x) \)
- Area of slice \( y = \) constant between \( c \) \& \( d \)

Idea: Volume is obtained by "summing" all the \( A(x) \)'s for \( a \leq x \leq b \)

Also,

\[ A(y) \)'s \quad \text{for} \quad c \leq y \leq d \]

More precisely, summing means integration.

**Theorem [Fubini]**

Fix \( f = f(x,y) : R \rightarrow \mathbb{R} \) a continuous function

Then, \( \iint_R f(x,y) \, dA \) exists (\( f \) is integrable on \( R \)) and we can compute it in 2 different **iterated ways**:

\[
\iint_R f(x,y) \, dA = \int_a^b \left( \int_c^d f(x,y) \, dx \right) \, dy = \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy
\]

**Note:** There are functions (not continuous!) when the 2 iterated integrals exist but the function is **NOT** integrable on \( R \).

- Often, one order of integration is easier than the other (Enough to do the easy one!)

**Proof idea:** The definition of \( \iint_R f(x,y) \, dA \) requires to use any partition, so we can make all \( \Delta y_k \) very small first.

Then, make \( \Delta x_k \) very small, and the limit gives \( \iint_R f(x,y) \, dA \).

\[ \Delta A_k = \Delta x_k \Delta y_k \]
Examples:

1. Compute \( \iiint_R (x+y) \, dA \) where \( R = [0,3] \times [1,4] \).

   The graph is a plane.

   \[
   f(x,y) = x + y \quad \text{is \textit{an} \underline{cunt}, so we use Fubini!} \quad \text{Fund Thm of Calc} \quad \text{each} \quad A(x) = A(y)
   \]

   \[
   \iiint_R f(x,y) \, dA = \int_0^3 \int_0^4 (x+y) \, dy \, dx = \int_0^3 \left[ \frac{xy}{2} + y^2 \right]_{y=1}^{y=4} \, dx
   \]

   \[
   = \int_0^3 \left( 4x + 16 - (x+1) \right) \, dx = \int_0^3 3x + 15 \, dx = \frac{3}{2} x^2 + 15x \bigg|_{x=0}^{x=3} = \frac{27}{2} + 45 = \frac{117}{2}
   \]

   \[
   \iiint_R f(x,y) \, dA = \int_0^4 \int_0^3 (x+y) \, dx \, dy = \int_0^4 \left[ \frac{x^2}{2} + 2xy \right]_{x=0}^{x=3} \, dy
   \]

   \[
   = \int_0^4 \left( \frac{9}{2} + 6y - 0 \right) \, dy = \left( \frac{9y}{2} + 3y^2 \right) \bigg|_{y=0}^{y=4} = \frac{36 + 48 - 9 + 3}{2} = \frac{117}{2}
   \]

2. Compute \( \iiint_R y \cos(xy) \, dA \) where \( R = [0,1] \times [0,\frac{\pi}{3}] \).

   \( f(x,y) = y \cos(xy) \) is \underline{continuous}, so we can use Fubini! [See if no order is easier than \underline{the other}]

   \[
   \int_0^{\pi/3} \int_0^1 y \cos(xy) \, dx \, dy = \int_0^{\pi/3} \int_0^1 \left( \frac{\sin(xy)}{y} \right) \, dx \, dy = \int_0^{\pi/3} \text{sin}y \, dy = \frac{\sin y}{y} \bigg|_0^{\pi/3} = \frac{\sqrt{3}}{2} - \frac{1}{2}
   \]

   The other order of integration is \underline{HARDER}!

   \[
   \int_0^1 \int_0^{\pi/3} y \cos(xy) \, dx \, dy = \int_0^1 \left( \frac{\sin xy}{y} - \frac{\sin xy}{x} \right) \, dy = \int_0^1 \left( \frac{\sin xy}{y} - \frac{\sin xy}{x} \right) \, dy
   \]

   \[
   = \left[ \frac{\pi}{3} \sin \frac{\pi}{3} x - 0 \right] + \left( \frac{\cos \frac{\pi}{3} x}{x^2} - \frac{1}{x^2} \right)
   \]

   \[
   \iiint_R f(x,y) \, dA = \int_0^{\pi/3} \int_0^1 \left( \frac{\pi}{3} \sin \frac{\pi}{3} x + \cos \frac{\pi}{3} x - \frac{1}{x^2} \right) \, dx
   \]

   Integration by parts:

   \[
   \lim_{x \to 0} \frac{-\text{cos}(\pi/3 \cdot x) - 1}{x} = \frac{\text{d}x}{x} \left( -\frac{\pi}{3} \sin \frac{\pi}{3} x + \frac{\cos \frac{\pi}{3} x}{X^2} - 1 \right) + \frac{1}{X} \left( \frac{\pi}{3} \sin \frac{\pi}{3} x + \frac{1}{X} \right) \bigg|_{x=0}^{x=1} = -\frac{\pi}{3} \sin \frac{\pi}{3} x + \frac{1}{X} \bigg|_{x=0}^{x=1} = \frac{2-\sqrt{3}}{3} + \frac{2}{3} = \frac{4}{3}
   \]