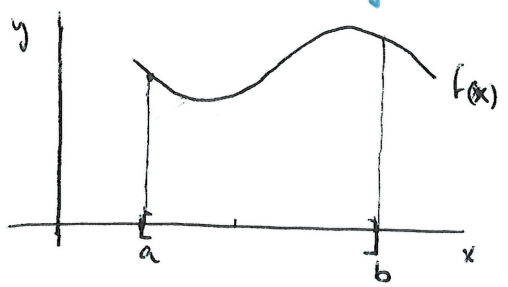
Lecture XXI (3/4/16) §14.1 Double integrals over Rectangular regions

§1 Recall: Integrals in one variable are used to compute signed areas under curves

We define them using Riemann sums:



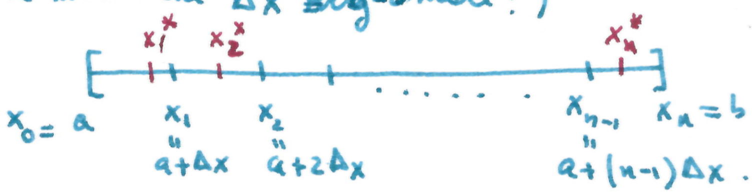
Area between the curve and the x-axis $= \int_a^b f(x)\,dx$

(Here: $f(x) \geq 0$ so signed area is usual area)

We approximate this area by covering it with rectangles

STEP 1: Break the interval $[a,b]$ into n intervals of equal length $\boxed{\Delta x = \frac{b-a}{n}}$

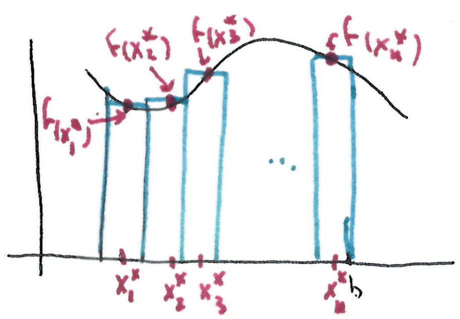
(We will make Δx very small!) This is called a REGULAR PARTITION.



x_k are grid points $k=1,\dots,n$

$x_k = a + k\Delta x$.

STEP 2: Pick a point x_k^* in $[x_{k-1}, x_k]$ for every $k = 1, \dots, n$.



For every k, we form the rectangle with base $[x_{k-1}, x_k]$ & height $= f(x_k^*)$ Call it rect_k

$$\boxed{\text{Area}(\text{rect}_k) = f(x_k^*)\,\Delta x.}$$

Def: The Riemann sum associated to this data is

$$\text{Sum of Areas of all rect}_k = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x =: \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

- We can pick the points x_k^* arbitrarily or following some rules:
  1. (1) Pick left pt of every interval, ie $x_k^* = x_{k-1}$ ⇝ LEFT RIEMANN SUM.
  2. (2) — right pt ————, ie $x_k^* = x_k$ ⇝ RIGHT ————.
  3. (3) — midpoint ————, ie $x_k^* = \frac{x_k + x_{k-1}}{2}$ ⇝ MIDPOINT ————.
- If x_k^* in $[x_{k-1}, x_k]$ is arbitrary, and $\Delta x_k = [x_{k-1}, x_k]$ is arbitrary, then we call the sum a GENERAL RIEMANN SUM

Definition:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k \qquad \Delta x = \max\{\Delta(x_k) : k = 1, \dots, n\}$$

We say f is integrable on $[a,b]$ whenever the (RHS) limit exists.

Theorem: If f continuous on $[a,b]$, then f is integrable on $[a,b]$.

§2 Double integrals: Fix $R = [a,b] \times [c,d] = \{(x,y) : \begin{smallmatrix} a \le x \le b \\ c \le y \le d \end{smallmatrix}\}$ rectangle in $\mathbb{R}^2$

and $f = \underset{(x,y)}{f} : R \longrightarrow \mathbb{R}$ where $f(x,y) \geq 0$

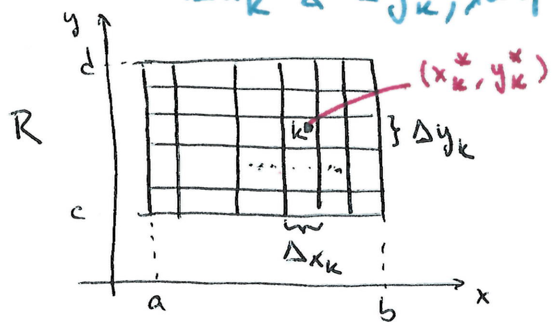
GOAL: Find the Volume of the solid bounded by the graph of f and R

(if $f(x,y)$ has arbitrary sign, we get a signed volume)

- We want to mimic what we did for functions of one variable, so we need 2 steps:

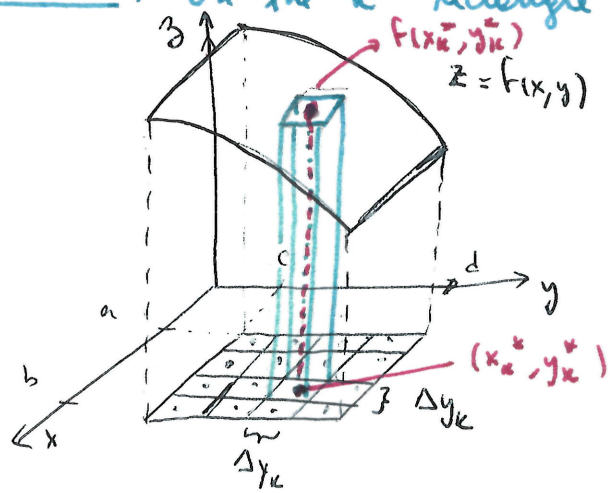
STEP 1: Break the rectangle R into N rectangular subregions with sides parallel to the x-axis & y-axis respectively. Their lengths of the k^{th} rectangle are Δx_k & Δy_k, respectively.

$$\boxed{\text{Area}(\text{rect}_k) = \Delta x_k \, \Delta y_k := \Delta A_k}$$



- The rectangles form a PARTITION of R
- The grid need not be regular (Δx_k can be different for different k's & the same for Δy_k's)
- We can order the rectangles (e.g., from left to right & bottom to top $\rightleftarrows$)

STEP 2: On the k^{th} rectangle, we pick any point (x_k^*, y_k^*)

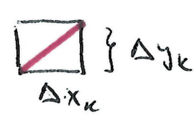


We build the k^{th} box with
- base = k^{th} rectangle
- height = $f(x_k^*, y_k^*)$

$$\boxed{\text{Volume}(k^{th} \text{ box}) = f(x_k^*, y_k^*) \, \Delta A_k}$$

$$\text{Sum}\left(\begin{smallmatrix}\text{Volumes of} \\ \text{all boxes}\end{smallmatrix}\right) = \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta x_k \, \Delta y_k .$$

- This sum approximates the Volume of the solid. The smaller the (diagonal of all) rectangles, the better the approximation (so, as $N \to +\infty$ / $\Delta = \text{diag} \to +\infty$ we should get the vol!)
- Write $\Delta = \max(\text{diag}) = \max\limits_{1 \le k \le N} \{ \sqrt{\Delta x_k^2 + \Delta y_k^2} \}$



Def: • $\displaystyle\iint_R f(x,y) \, dA := \lim_{\Delta \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta A_k$ (double integral of f over R)

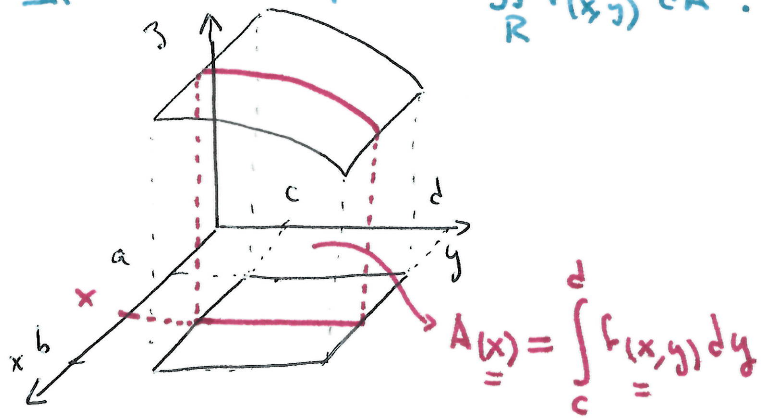
- f is integrable on R if the limit exists for all partitions of R and all choices of (x_k^*, y_k^*) within those partitions.

Note: If $f(x,y) \geq 0$, the integral is the Volume bounded by R and the graph of f.

• If f has arbitrary sign, we get a signed volume, also called <u>net volume</u> (the parts where $f(x,y) \leq 0$ contribute negative volume)

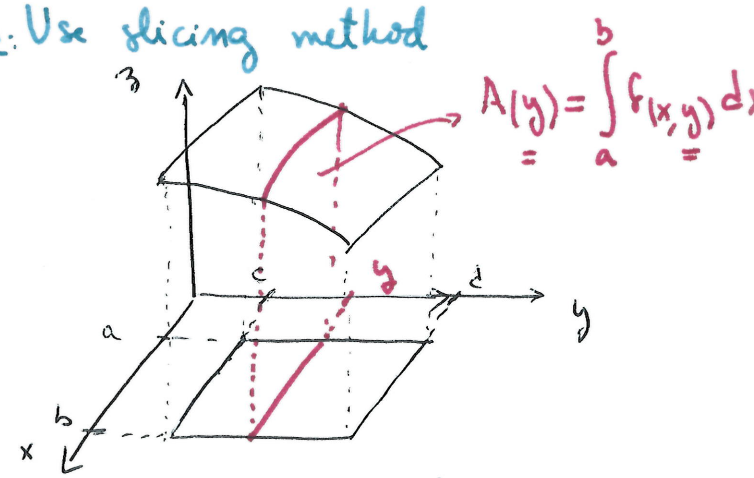
§3 Iterated integrals:

<u>Q</u>: How to compute $\iint_R f(x,y)\, dA$? <u>A</u>: Use slicing method



slice along yz-planes
x = constant between a & b

Cross sectional area = $A(x)$



slice along xz-planes
y = constant between c & d

cross sectional area = $A(y)$

<u>Idea</u>: Volume is obtained by "summing" all the $A(x)$'s for $a \leq x \leq b$

Also, ——————————————— $A(y)$'s for $c \leq y \leq d$

More precisely: summing means integration.

<u>Theorem</u> [FUBINI] Fix $f = f(x,y) : R \longrightarrow \mathbb{R}$ a <u>continuous</u> function

Then, $\iint_R f(x,y)\, dA$ exists (f is <u>integrable on R</u>) and we can compute it in 2 different <u>iterated ways</u>:

$$\iint_R f(x,y)\, dA = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\, dx}_{=A(y)} \Big)\, dy = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\, dy}_{=A(x)} \Big)\, dx.$$

<u>Note</u>: • There are functions (not continuous!) where the 2 iterated integrals exist but the function is NOT integrable over R

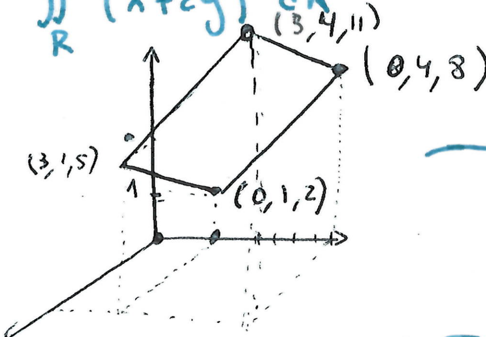
• Often, one order of integration is easier than the other! (Enough to do the easy one!)

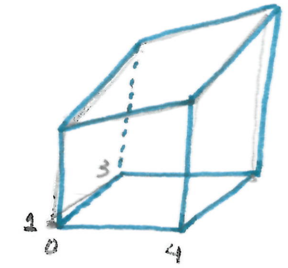
<u>Proof idea</u>: The definition of $\iint_R f(x,y)\, dA$ requires to use <u>any</u> partitions, so we can make all Δy_k very small first $\sum_k f(x_k^*, y_k^*)\, \Delta A_k \cong \sum_k \left(\int_c^d f(x_k^*, y)\, dy \right) \Delta x_k$.

Then, make Δx_k very small, and the limit gives $\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$. $[\Delta A_k = \Delta x_k \, \Delta y_k]$

Examples: ① Compute $\iint_R (x+2y)\,dA$ $\qquad R=[0,3]\times[1,4]$

Graph is a plane





$f(x,y)=x+2y$ is cont, ~~so~~ we use Fubini: & Fund Thm of Calculus to get each A(x) & A(y)

• $\int_0^3\left(\int_1^4 f(x,y)\,dy\right)dx = \int_0^3\left(\int_1^4 x+2y\,dy\right)dx = \int_0^3 \left(xy+y^2\Big|_{y=1}^{y=4}\right)dx$

$= \int_0^3 \left(4x+16-(x+1)\right)dx = \int_0^3 3x+15\,dx = \frac{3}{2}x^2+15x\Big|_{x=0}^{x=3} = \frac{27}{2}+45 = \boxed{\frac{117}{2}}$

• $\int_1^4\left(\int_0^3 f(x,y)\,dx\right)dy = \int_1^4\left(\int_0^3 x+2y\,dx\right)dy = \int_1^4 \frac{x^2}{2}+2yx\Big|_{x=0}^{x=3} dy$

$= \int_1^4 \left(\frac{9}{2}+6y-0\right)dy = \left(\frac{9}{2}y+3y^2\right)\Big|_{y=1}^{y=4} = \frac{36}{2}+48-\left(\frac{9}{2}+3\right) = 63-\frac{9}{2} = \boxed{\frac{117}{2}}$

② Compute $\iint_R y\cos(xy)\,dA$ $\qquad R=[0,1]\times[0,\frac{\pi}{3}]$

$f(x,y)=y\cos(xy)$ is continuous, so we can use Fubini! [See if one order is easier than the other]

$y(\cos xy) \rightsquigarrow y=\frac{\partial(xy)}{\partial x}$ use dx & then dy!

• $\int_0^{\pi/3}\left(\int_0^1 y\cos(xy)\,dx\right)dy = \int_0^{\pi/3}\left(\int_0^1 \frac{d}{dx}(\sin xy)\,dx\right)dy = \int_0^{\pi/3} \sin xy\Big|_{x=0}^{x=1} dy$

$= \int_0^{\pi/3} \sin y - \underset{0}{\underset{\shortparallel}{\sin 0}}\,dy = \int_0^{\pi/3} \sin y\,dy = -\cos y\Big|_{y=0}^{y=\pi/3} = \left(-\frac{\sqrt{3}}{2}-(-1)\right) = \boxed{\frac{2-\sqrt{3}}{2}}$

• The other order of integration is MUCH HARDER!

$\int_0^{\pi/3} y\cos(xy)\,dy = ?$ $\qquad$ Write $h(y)=y$ $\qquad$ [derivative = derivative in y]

$g'(y)=\cos xy \rightsquigarrow g(y)=\frac{\sin(xy)}{x}$ & use integration by parts!

$= \int_0^{\pi/3} (hg)'-h'g\,dy = \int_0^{\pi/3}\left(y\frac{\sin xy}{x}\right)'-\frac{\sin xy}{x}\,dy = \frac{y\sin xy}{x}\Big|_{y=0}^{\pi/3} - \left(\frac{-\cos xy}{x^2}\right)\Big|_{y=0}^{y=\pi/3}$

$= \left(\frac{\pi}{3x}\sin\frac{\pi}{3}x - 0\right) + \left(\frac{\cos\frac{\pi}{3}x}{x^2}-\frac{1}{x^2}\right)$

$\iint_R f(x,y)\,dA = \int_0^1 \left(\underbrace{\frac{\pi}{3x}\sin\frac{\pi}{3}x + \frac{\cos\frac{\pi}{3}x}{x^2}}-\frac{1}{x^2}\right)dx$ can be computed via integration by parts!

$= \left(\frac{d}{dx}\left(-\frac{1}{x}\cos\frac{\pi}{3}x\right)-\frac{1}{x^2}\right)$, so $= \frac{-1}{x}\cos\frac{\pi}{3}x+\frac{1}{x}\Big|_{x=0}^{x=1} = \left(-\cos\frac{\pi}{3}+1\right)$

$-\lim_{x\to 0}\frac{-(\cos(\frac{\pi}{3}x)-1)}{x} = -\frac{\sqrt{3}}{2}+1+\lim_{x\to 0}-(\sin\frac{\pi}{3}x)\cdot\frac{\pi}{3} = \boxed{\frac{2-\sqrt{3}}{2}}$