Lecture XXI: 8.15.2 Double integrals on general regions

Last Time: Defined integrals on rectangular regions via Riemann sums:

\[
\Delta x_k, \Delta y_k = \Delta A_k
\]

mid of \( N \) rectangles \( A_{\text{area}}(\text{Rect}_k) = \Delta x_k \Delta y_k = \Delta A_k \)

Pick \( (x^*_k, y^*_k) \) in \( \text{Rect}_k \) and \( \Delta = \max \left\{ \frac{\Delta x_k^2 + \Delta y_k^2}{k \in N} \right\} \)

Box: Base = \( \text{Rect}_k \) & Height = \( f(x^*_k, y^*_k) \)

approximates the solid bounded by \( R \) & the graph of \( f \)

Defined: \( \iint_R f(x,y) \, dA = \lim_{\Delta \to 0} \sum_{k=1}^N f(x^*_k, y^*_k) \Delta A_k \)

(Take over all partitions of \( R \) & choice of pts \( (x^*_k, y^*_k) \))

Fubini's Theorem: If \( f \) is continuous on \( R \),

then \( f \) is integrable on \( R \) and we can compute it in 2 iterated ways:

\[
\iint_R f(x,y) \, dA = \left( \int_a^b \left( \int_c^d f(x,y) \, dx \right) \, dy \right) = \left( \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy \right)
\]

\[
= A(y) =: \int_c^d f(x,y) \, dy = \text{Area under the curve } f(x,y) \text{ [y fixed]}
\]

Remarks: Some orders are easier than others,

Limits of integration are independent

Today: Use the same ideas to integrate on general regions.

\[ f: \mathbb{R} \to \mathbb{R} \text{, } R \text{ region in } \mathbb{R}^2 \text{ closed & bounded } \]

A collection of \( (\text{Rect}_k)_{k=1}^N \)

Approximate \( R \) by rectangles \( \text{inside} \) \( R \) with sides parallel to \( x \)-\( y \)-axes, of length \( \Delta x_k \& \Delta y_k \), max.

(\( \Delta \) the smaller the diagonal, the more rectangles, we can put!)

Pick a point \( (x^*_k, y^*_k) \) in \( \text{Rect}_k \), \( A_{\text{area}}(\text{Rect}_k) = \Delta A_k \)

\( \Delta = \max \left\{ \frac{\Delta x_k^2 + \Delta y_k^2}{k \in N} \right\} \) max diagonal

Boxes: Base = \( \text{Rect}_k \) & Height = \( f(x^*_k, y^*_k) \)

Net Volume of the solid bounded by \( R \) & the graph of \( f \) equals

\[ \iiint_R f(x,y,z) \, dV = \lim_{\Delta \to 0} \sum_{k=1}^N \left( \frac{f(x^*_k, y^*_k, z^*_k)}{\Delta A_k} \right) \]

& \( \text{vol} (\Box_k) \)
3.2 Iterated integrals: We will decide the order of integration of $f: \mathbb{R} \to \mathbb{R}$ based on the nature of $R$ [Slicing method depends on $R$].

**Type I**

For Type I regions, the lower & upper bounds in the $y$-direction are graphs of 2 continuous functions $g(x)$ & $h(x)$, respectively. The slicing method yields:

$$A(x) = \int_{g(x)}^{h(x)} f(x,y) \, dy$$

**Conclusion:** 
(Fubini for Type I) 
$$\iint R f(x,y) \, dA = \int_a^b \left( \int_{g(x)}^{h(x)} f(x,y) \, dy \right) \, dx$$ 

**Example:** Compute $\iint R x^2 y \, dA$

**Step 1:** Draw $R$ to see if it's Type I. (bounded by $g(x)=3x^2$ & $h(x)=16-x^2$)

**Step 2:** Find the intersection points of the 2 graphs:

$$3x^2 = 16-x^2 \implies 4x^2 = 16 \implies x = \pm 2$$

**Step 3:** 
$$\iint R f(x,y) \, dA = \int_{-2}^{2} \left( \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} x^2 y \, dy \right) \, dx$$
\[
\mathcal{R} = \int_{-2}^{c} x^2 y^2 \, dx = \int_{y=-3x^2}^{y=16-x^2} x^2 (16-x^2-3x^2) \, dx \\
= \int_{-2}^{c} \frac{x^2}{2} \left(16 - 8x^4 - 32x^2\right) \, dx = \int_{-2}^{c} \left(16x^2 - 4x^6 - 16x^4\right) \, dx \\
= \frac{16}{c} \left[ x^3 - \frac{4}{7} x^7 - \frac{16}{5} x^5 \right]_{x=-2}^{x=c} = \frac{69632}{210} = \frac{34816}{105}
\]

**Type II**

Region \( \mathcal{R} \) is bounded on the left and right by the graphs of two continuous functions \( x = g(y) \) and \( x = h(y) \), respectively, and \( c \leq y \leq d \).

Using y-slices: 
\[
A(y) = \int g(y) h(y) \, dy
\]

Conclusion: 
\[
\iint_{R} f(x,y) \, dA = \int_{c}^{d} \left( \int_{g(y)}^{h(y)} f(x,y) \, dx \right) \, dy
\]

For \textbf{Type II Regions}

Definition: If \( R \) is a general region, we cannot switch the order of integration.

If \( R \) is both \textbf{Type I and II}, we have 2 ways of computing but the limits of integration are different.

Example: \( R = \frac{y}{x} = x^2 \) or \( \frac{y}{x} = \frac{1}{2} \)

\[
\iint_{R} f(x,y) \, dA = \int_{0}^{1} \left( \int_{0}^{\frac{1}{2}} f(x,y) \, dy \right) \, dx
\]

Now examples: HW 7 & Recitation 8

### §3 Decomposition of Regions

Write \( R = R_1 \cup R_2 \) (divide \( R \) into nonoverlapping regions).

Then: 
\[
\iint_{R} f(x,y) \, dA = \iint_{R_1} f(x,y) \, dA + \iint_{R_2} f(x,y) \, dA
\]
We can use this to integrate over regions that are not of Type I or Type II but that we can decompose into regions of these types without overlapping.

Eg: \( R = \{(x, y) : -2 \leq x, y \leq 2, x^2 + y^2 \geq 1\} \)

\[ \begin{align*}
\text{Type I} & \quad \text{Type II} \\
T: & \quad \left\{ \begin{array}{l}
y = g(x) = 0 \\
y = h(x) = x \\
1 \leq x \leq 2
\end{array} \right. & \left\{ \begin{array}{l}
y = g(x) = |1-x^2| \\
y = h(x) = 2 \\
0 \leq x \leq 1
\end{array} \right.
\end{align*} \]

\[ \begin{align*}
T: & \quad \left\{ \begin{array}{l}
y = g(y) = 1 \\
x = h(y) = 2 \\
0 \leq y \leq 2
\end{array} \right.
\end{align*} \]

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2 continuous functions \( f = f(x, y) : R \rightarrow \mathbb{R} \)

\[ g = g(x, y) : R \rightarrow \mathbb{R} \]

where \( f(x, y) \geq g(x, y) \) for all \((x, y)\) in \( R \)

Then, the volume of the solid bounded by the graphs of \( f \) and \( g \) equals:

\[ \text{Vol} = \iint_{R} (f(x, y) - g(x, y)) \, dA = \iint_{R} f(x, y) \, dA - \iint_{R} g(x, y) \, dA \]

Special case: \( g(x, y) = 0 \) \& \( f(x, y) = 1 \). Solid \( S \)

\[ \text{Vol} = \iiint_{R} 1 \, dV \]

Prove: \( \text{Area} (R) = \text{Area} (R) \cdot 1 = \iint_{R} f(x, y) \, dA = \iint_{R} 1 \, dA \).