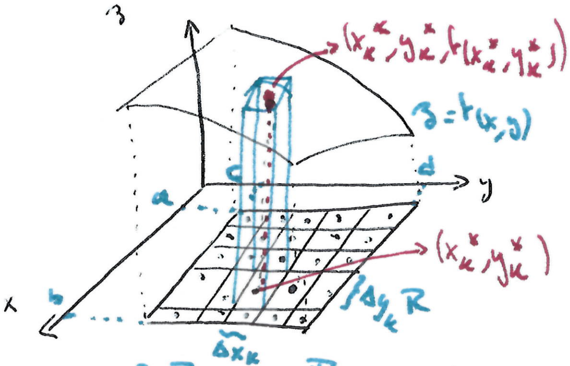
Lecture XXII: §14.2 Double integrals over General Regions

Last Time: Defined integrals over rectangular regions via Riemann sums:



$f : R \longrightarrow \mathbb{R}$

$R = [a,b] \times [c,d]$

grid of N rectangles $\text{Area}(\text{Rect}_k) = \Delta x_k \Delta y_k = \Delta A_k$

Pick (x_k^*, y_k^*) in Rect_k & $\Delta = \max\limits_{1 \le k \le N} \left\{ \sqrt{\Delta x_k^2 + \Delta y_k^2} \right\}$ (small)

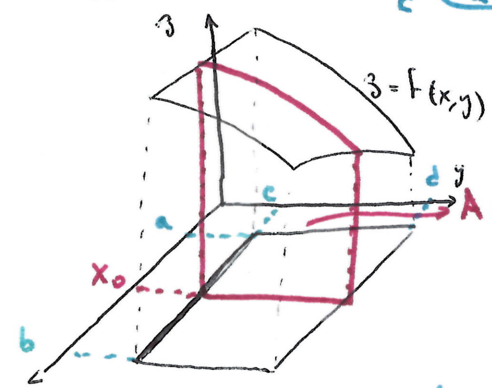
Box: Base $= \text{Rect}_k$ & Height $= f(x_k^*, y_k^*)$.
approximates the solid bounded by R & the graph of f

Defined: $\displaystyle\iint_R f(x,y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\, \Delta A_k$

(limit over ALL partitions of R & choices of pts (x_k^*, y_k^*))

Fubini's Theorem: If f is continuous on R, then f is integrable over R and we can compute it in 2 iterated ways:

$$\iint_R f(x,y)\, dA = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\, dx}_{=:\,A(y)} \Big)\, dy = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\, dy}_{=:\,A(x)} \Big)\, dx$$

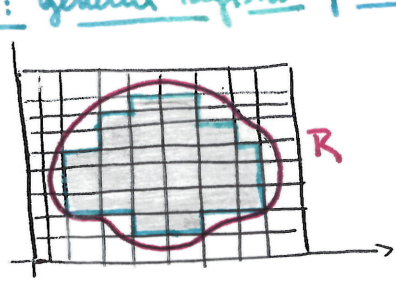


$A(\underline{\underline{x}}) = \displaystyle\int_c^d f(x_0, y)\, dy$ = Area under the curve $f(x_0, y)$ [x_0 fixed]

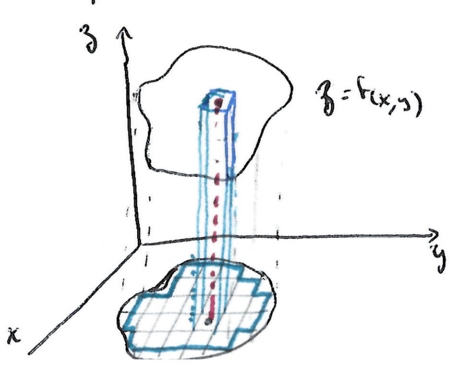
Remarks:
- Some orders are easier than others.
- Limits of integration are INDEPENDENT

TODAY Use the same ideas to integrate over GENERAL regions
$f : R \longrightarrow \mathbb{R}$, R region in $\mathbb{R}^2$ closed & bounded

§1: General regions of integration:



- Approximate R by a collection of $(\text{Rect}_k)_{k=1}^N$ rectangles INSIDE R with sides parallel to x- & y-axes, of length Δx_k & Δy_k, resp.
  (The smaller the diagonal, the more rectangles we can fit!)
- Pick a point (x_k^*, y_k^*) in Rect_k ; $\text{Area}(\text{Rect}_k) = \Delta A_k = \Delta x_k \Delta y_k$
- $\Delta := \max\limits_{1 \le k \le N} \left\{ \sqrt{\Delta x_k^2 + \Delta y_k^2} \right\}$ max diagonal



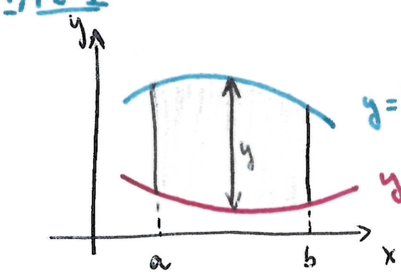
Boxes: Base $= \text{Rect}_k$ & height $= f(x_k^*, y_k^*)$

Net Volume of the solid bounded by R & the graph of f equals

Def: $\displaystyle\iint_R f\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{N} \big(\underbrace{f(x_k^*, y_k^*)\, \Delta A_k}_{= \text{Vol}(\text{Box}_k)} \big)$ [limit over all partitions & choices of points]

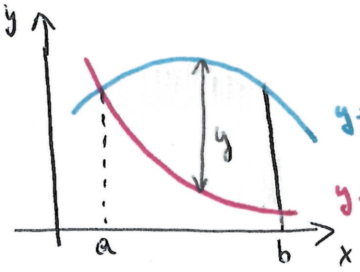
§2 Iterated integrals: We will decide the order of integration of $f: R \to \mathbb{R}$ cont. based on the nature of R [Slicing method depends on R]

, TYPE I



$g(x) \le y \le h(x)$

$a \le x \le b$

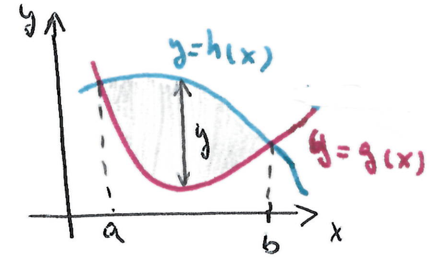


$g(x) \le y \le h(x)$

$a \le x \le b$

$\boxed{g(a) = h(a)}$

(2 options = meet at $x=a$ or $x=b$)

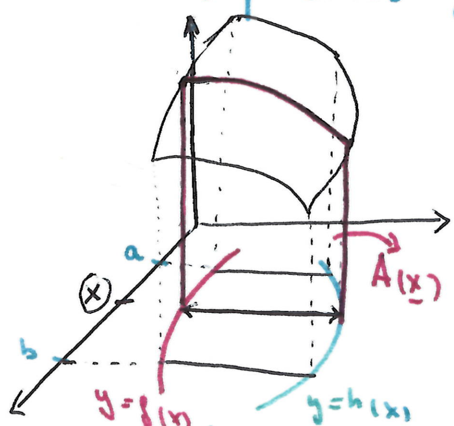


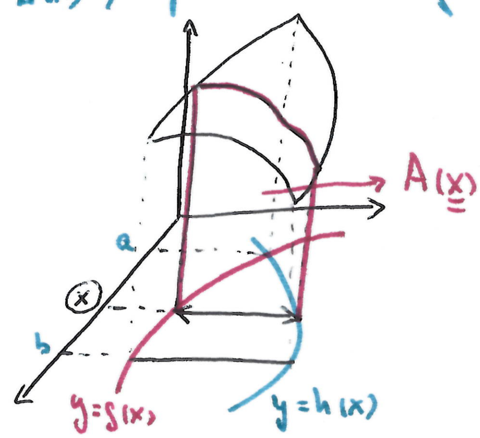
$g(x) \le y \le h(x)$

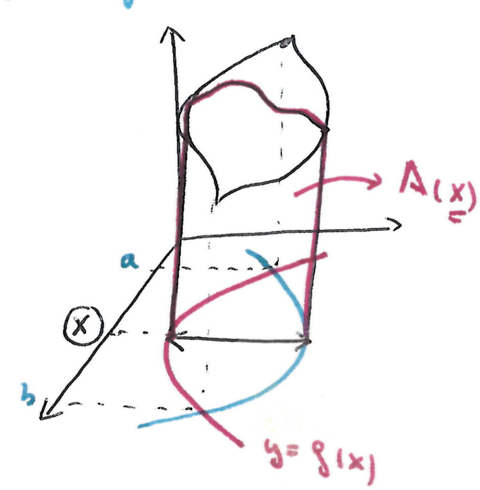
$a \le x \le b$

$\boxed{\begin{array}{l} g(a) = h(a) \\ g(b) = h(b) \end{array}}$

For Type I regions, the lower & upper bounds in the y-direction are graphs of 2 continuous functions $g(x)$ & $h(x)$, resp. The slicing method yields:





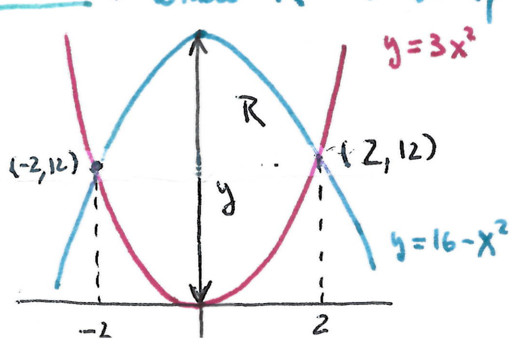


$A(\underline{x}) = \int_{g(x)}^{h(x)} f(\underline{x}, y)\, dy$ is the area under the graph of $f(\underline{x}, y) : [g(\underline{x}), h(\underline{x})] \to \mathbb{R}$ (x fixed, y varies)

Conclusion: (Fubini for Type I) $\displaystyle \iint_R f(x,y)\, dA = \int_a^b \underbrace{\left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right)}_{=: A(x)} dx$ for TYPE I Regions

Example: Compute $\iint_R x^2 y\, dA$ R: region bounded by $y = 3x^2$ & $y = 16 - x^2$.

STEP I: Draw R to see if it's TYPE I. (bounded by $g(x) = 3x^2$ & $h(x) = 16 - x^2$) ✓



STEP II: Find the intersection points of the 2 graphs:

$3x^2 = 16 - x^2 \iff 4x^2 = 16 \iff \boxed{x = \pm 2}$

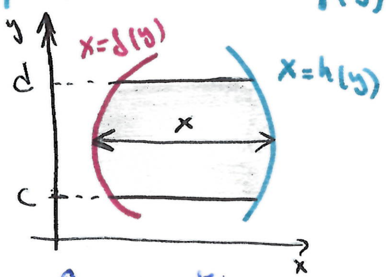
STEP III: $\displaystyle \iint_R f(x,y)\, dA = \int_{-2}^{2} \left(\int_{3x^2}^{16-x^2} x^2\, y\, dy \right) dx$ ↳ x is constant inside!

$$= \int_{-2}^{2} \frac{x^2y^2}{2}\Bigg|_{y=3x^2}^{y=16-x^2} dx = \int_{-2}^{2} \frac{x^2}{2}\left((16-x^2)^2-(3x^2)^2\right)dx$$

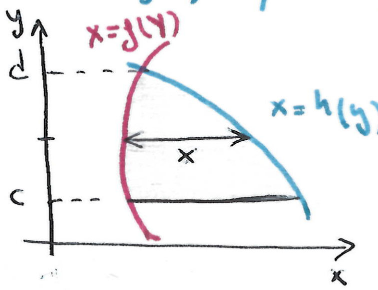
$$= \int_{-2}^{2} \frac{x^2}{2}\left(16^2-8x^4-32x^2\right)dx = \int_{-2}^{2}\left(\frac{16^2}{2}x^2-4x^6-16x^4\right)dx$$

$$= \frac{16^2}{6}x^3-\frac{4}{7}x^7-\frac{16}{5}x^5\Bigg|_{x=-2}^{x=2} = \frac{69632}{210} = \boxed{\frac{34816}{105}}$$

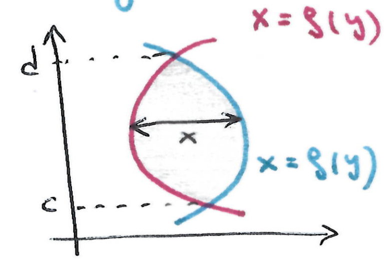
- **TYPE II** R bounded on the left & right by the graph of 2 continuous functions $x=g(y)$ & $x=h(y)$, respectively and $c\le y\le d$.



$g(y)\le x\le h(y)$
$c\le y\le d$



$g(y)\le x\le h(y)$
$c\le y\le d$
$\boxed{g(d)=h(d)}$



$g(y)\le x\le h(y)$
$c\le y\le d$
$\boxed{\begin{array}{l} g(c)=h(c)\\ g(d)=h(d)\end{array}}$

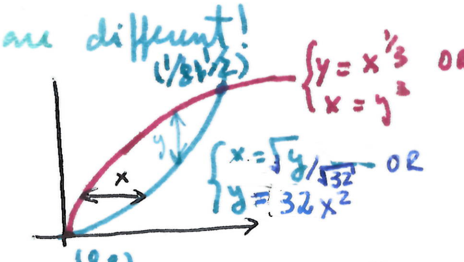
(2 options meet at $y=c$ or $y=d$)

Use y-slices: $A(\underline{y}) = \displaystyle\int_{g(y)}^{h(y)} f(x,\underline{y})\,dx$ is the area under the graph of $f(x,\underline{y}):[g(\underline{y}),h(\underline{y})]\to\mathbb{R}$ (y fixed, x varies)

Conclusion: (FUBINI for TYPE II) $\displaystyle\iint_R f(x,y)\,dA = \int_c^d\left(\int_{g(y)}^{h(y)} f(x,y)\,dx\right)dy$ for TYPE II Regions

Warning: If R is general region, we cannot switch the order of integration.
If R is both TYPE I or II we have 2 ways of computing but the limits of integration are different!

Eg. R:



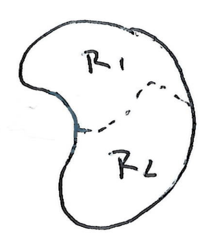
$$\iint_R f(x,y)\,dA = \int_0^{1/8}\left(\int_{32x^2}^{x^{1/3}} f(x,y)\,dy\right)dx$$

$$= \int_0^{1/2}\left(\int_{y^3}^{\sqrt{y}/\sqrt{32}} f(x,y)\,dx\right)dy$$

- More examples: HW 7 & Recitation 8.

§3 Decomposition of Regions

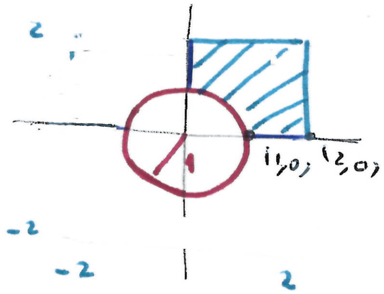
Write $R = R_1\cup R_2$ (divide R into 2 NONOVERLAPPING reg.)

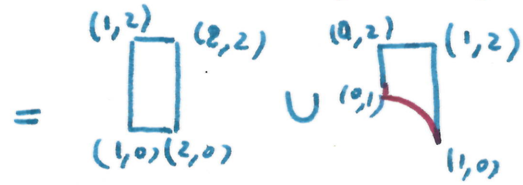


Then: $\boxed{\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA}$

We can use this to integrate over regions that are not of Type I nor Type II but that we can decompose into non overlapping regions of these types

Eg: $R = \{ (x,y) : 0 \leq x, y \leq 2 \ , \ x^2+y^2 \geq 1 \}$



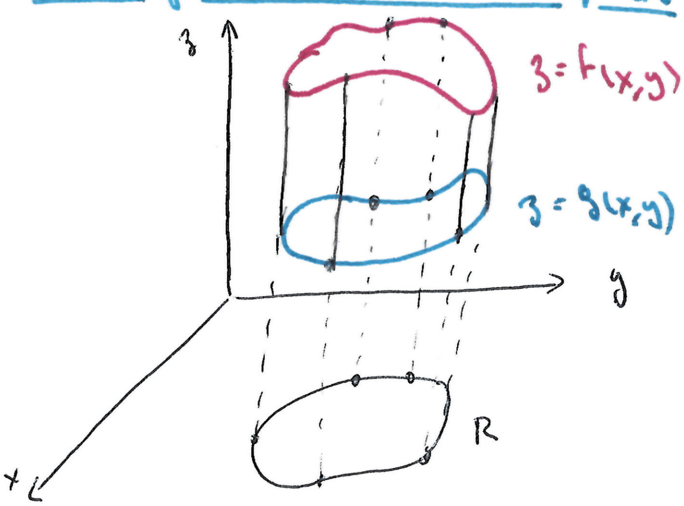


Type I & II — Type I.

$\underline{\underline{\mathrm{I}}}$: $\begin{cases} y = g(x) = 0 \\ y = h(x) = 2 \\ 1 \leq x \leq 2 \end{cases}$ $\qquad$ $\begin{cases} y = g(x) = \sqrt{1-x^2} \\ y = h(x) = 2 \\ 0 \leq x \leq 1 \end{cases}$

$\underline{\underline{\mathrm{II}}}$: $\begin{cases} x = g(y) = 1 \\ x = h(y) = 2 \\ 0 \leq y \leq 2 \end{cases}$

§4 Regions between two surfaces



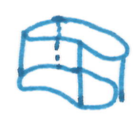
2 continuous functions $f = f(x,y) : R \longrightarrow \mathbb{R}$

$g = g(x,y) : R \longrightarrow \mathbb{R}$

where $f(x,y) \geq g(x,y)$ for all (x,y) in R

Then, the VOLUME of the solid bounded by the graphs of f & g equals:

$$\mathrm{Vol} = \iint_R (f(x,y) - g(x,y))\, dA = \iint_R f(x,y)\, dA - \iint_R g(x,y)\, dA$$

Special case: $g(x,y) = 0$ & $f(x,y) = 1$. solid = S  $\}$ ht = 1

Prop: $\boxed{\mathrm{Area}(R)} = \mathrm{Area}(R) \cdot \underset{\text{height}}{\underset{\shortparallel}{1}} = \mathrm{Vol}(S) = \iint_R f(x,y)\, dA = \boxed{\iint_R 1\, dA}$.