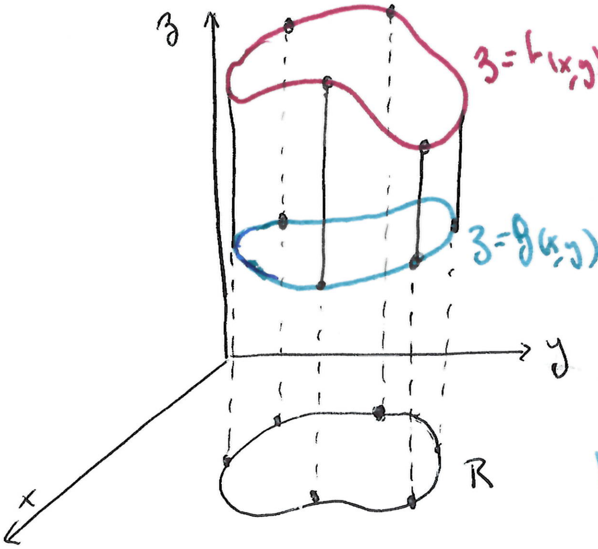
Lecture XXIII: §14.3 Double Integrals in polar coordinates

§14.1 Regions between 2 surfaces



& cont. functions $f=f(x,y): R \longrightarrow \mathbb{R}$

$g = g(x,y) : R \longrightarrow \mathbb{R}$

where $f(x,y) \geq g(x,y)$ for all (x,y) in R

Then, the VOLUME of the solid bounded by the graphs of f & g equals:

$$\mathrm{Vol} = \iint_R (f(x,y) - g(x,y))\, dA = \iint_R f(x,y)\, dA - \iint_R g(x,y)\, dA$$

Special case: $g(x,y) = 0$ & $f(x,y) = 1$ Solid $= S$

Prop: $\boxed{\mathrm{Area}(R)} = \mathrm{Area}(R) \cdot \underbrace{1}_{ht=1} = \mathrm{Vol}(S) = \iint_R f(x,y)\, dA = \boxed{\iint_R 1\, dA}$

§14.3 Double integrals in polar coordinates

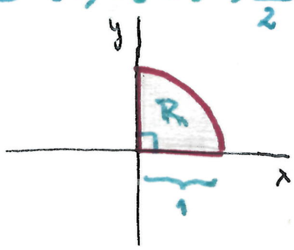
Conversion rules: $\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \end{cases} \longleftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$ (sign of y & x determine the value of θ)

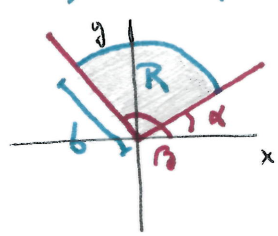
$0 \leq r$, $0 \leq \theta \leq 2\pi$

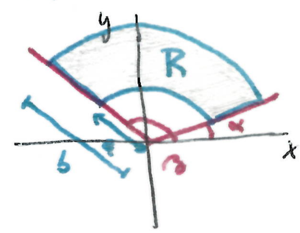
§1. Polar rectangles

Polar rectangle $= R = \{(r,\theta)\ 0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta\} = [a,b] \times [\alpha, \beta]$

Eg: ① $[0,1] \times [0, \frac{\pi}{2}]$ ② $[0,b] \times [\alpha,\beta]$ ③ $[a,b] \times [\alpha,\beta]$



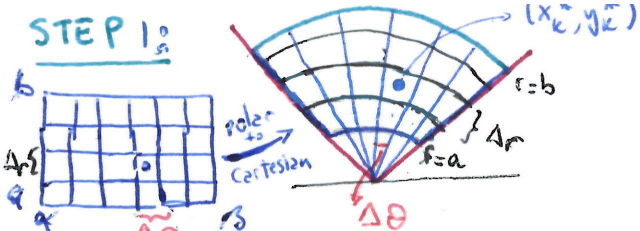




GOAL: Given $f_{(x,y)}$ continuous on R, compute $\iint_R f(x,y)\, dA$ using polar coordinates (because R is a rectangle in polar coordinates but not in cartesian coordinates.

- As usual, we define it by Riemann sums: 1) partition of R into rectangles & one point on each piece
  2) Way to compute areas of polar rectangles

STEP 1:



1) Divide $[a,b]$ into M rectangles of length $\Delta r = \frac{b-a}{M}$

2) Divide $[\alpha,\beta]$ " N ———— $\Delta\theta = \frac{\beta-\alpha}{N}$

3) Consider the rectangles obtained by these 2 partitions

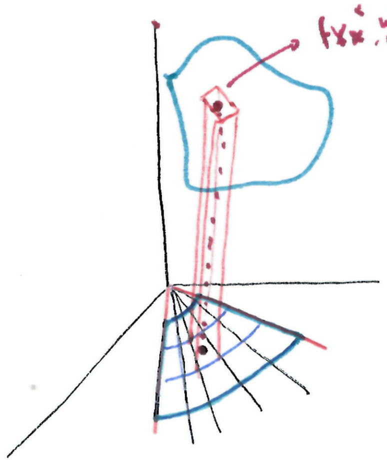
$R_{kj} :=$ rectangles $= [r_{k-1}, r_k] \times [\theta_{j-1}, \theta_j]$, $k = 1, \dots, M$, $j = 1, \dots, N$

$$\begin{cases} ① \text{ gives } r_0 = a < r_1 = a + \Delta r < \dots < r_{M-1} = a + (M-1)\Delta r < r_M = b \\ ② \text{ gives } \theta_0 = \alpha < \theta_1 = \alpha + \Delta\theta < \dots < \theta_{N-1} = \alpha + (N-1)\Delta\theta < \theta_N = \beta \end{cases}$$

$\boxed{n = M \cdot N}$

STEP 2: On every rectangle $Rect_k$, we pick any point $(x_k^*, y_k^*) = (r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)$

Take Boxes of base $= Rect_k$ & height $f(x_k^*, y_k^*)$ & add their volume



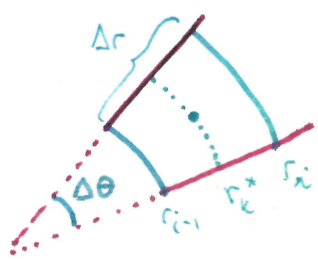
$$V = \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

$\Delta A_k =$ Area of $Rect_k$ $\qquad \Delta := \max\{\Delta r, \Delta\theta\}$

By definition: (*) $\boxed{\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \iint_R f(x,y)\, dA}$

Q: How to compute ΔA_k? As in §14.1, we want to express it in terms of Δr & $\Delta\theta$.

Since we can pick (x_k^*, y_k^*) freely, let's pick it so r_k^* is the midpoint of $[r_{i-1}, r_i]$



so $r_{i-1} := r_k^* - \frac{\Delta r}{2}$

$r_i := r_k^* + \frac{\Delta r}{2}$

$\boxed{\Delta A_k}$ = Area  − Area 

$$= \frac{\pi r_i^2 \cdot \Delta\theta}{2\pi} - \frac{\pi r_{i-1}^2\, \Delta\theta}{2\pi}$$

$$= \frac{\Delta\theta}{2}\left(r_i^2 - r_{i-1}^2\right) = \frac{\Delta\theta}{2}\left(r_{i-1} + r_i\right)\left(r_i - r_{i-1}\right)$$

$$= \frac{\Delta\theta}{2}\left(2 r_k^*\right)\left(2 \frac{\Delta r}{2}\right) = \boxed{r_k^*\, \Delta\theta\, \Delta r}$$

Replace this in (*)

$$\iint_R f(x,y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^* \cos\theta_k^*, r_k^* \sin\theta_k^*)\, r_k^*\, \Delta r\, \Delta\theta \overset{\text{def.}}{=} \iint_R f(r\cos\theta, r\sin\theta)\, r\, dA_{r\theta}$$

__Theorem:__ f continuous on R = polar rectangle $[a,b] \times [\alpha,\beta]$. Then f is integrable on R &

$$\iint_R f(x,y)\, dA = \int_a^b \left(\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, \underline{r}\, d\theta \right) dr = \int_\alpha^\beta \left(\int_a^b f(r\cos\theta, r\sin\theta)\, \underline{r}\, dr \right) d\theta$$

Example Find the volume of the region beneath the surface $z = xy + 5$ and above the annular region $R = \{(r,\theta) : 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

Soln: Use the Theorem!

$$f(x,y) = f(r\cos\theta, r\,\text{sen}\,\theta) = (r\cos\theta)(r\,\text{sen}\,\theta) + 5 = r^2\cos\theta\,\text{sen}\,\theta + 5$$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \qquad = \frac{r^2}{2}\left(2\,\text{sen}\,\theta\cos\theta\right) + 5 = \boxed{\frac{r^2}{2}(\text{sen}\,2\theta) + 5}$$

$$\text{Vol} = \int_0^{2\pi}\left(\int_2^4 \left(\frac{r^2}{2}(\text{sen}\,2\theta) + 5\right) r\,dr\right) d\theta = \int_0^{2\pi}\left(\int_2^4 \frac{r^3}{2}\text{sen}\,2\theta + 5r\,dr\right) d\theta$$

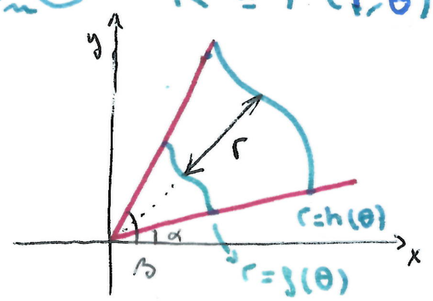
$$= \int_0^{2\pi} \left.\frac{r^4}{8}\text{sen}\,2\theta + \frac{5}{2}r^2\right|_{r=2}^{r=4} d\theta = \int_0^{2\pi}(30\,\text{sen}\,2\theta + 30)\,d\theta = \left(-15\cos 2\theta + 30\theta\right)\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \boxed{60\pi}$$

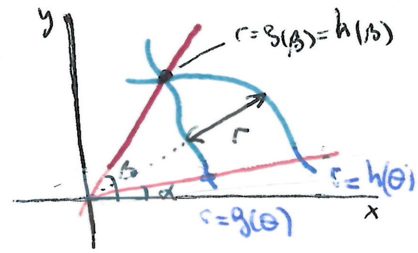
§2 General polar regions: (f continuous on R)

From last time, we know there are 2 types of regions over which we can integrate:

• **Type I** $R = \{(r,\theta) : \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta)\}$ $[0 \le \beta - \alpha \le 2\pi]$

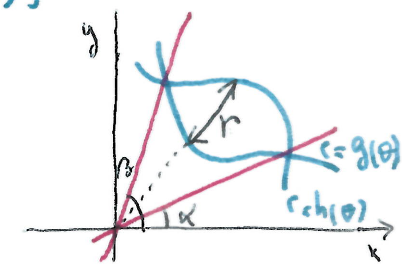


$\boxed{\text{no restriction on } \alpha < \beta}$.



$\boxed{g(\beta) = h(\beta),\ \alpha \le \beta}$

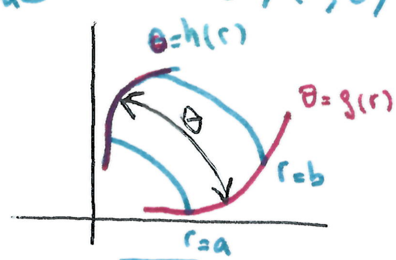
[twin picture: $g(\alpha) = h(\alpha)$, $\beta \ge \alpha$]



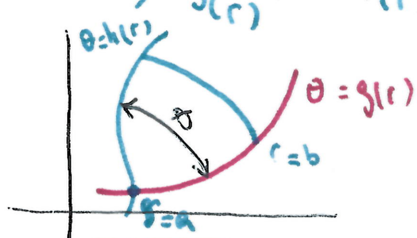
$\boxed{\begin{array}{l} g(\alpha) = h(\alpha) \\ g(\beta) = h(\beta) \end{array}}$

Theorem: $$\iint_R f(x,y)\,dA = \int_\alpha^\beta\left(\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\right)d\theta \quad \text{for Type I regions}$$

• **Type II** $R = \{(r,\theta) : a \le r \le b,\ g(r) \le \theta \le h(r)\}$ $[0 \le a < b]$

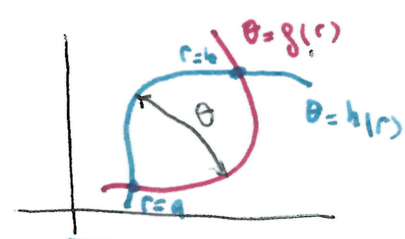


$\boxed{a<b}$



$\boxed{g(a) = h(a)\ ;\ a<b}$

[twin picture $g(b) = h(b)$; $a<b$]



$\boxed{\begin{array}{l} g(a) = h(a) \\ g(b) = h(b) \end{array}}$

Theorem: $$\iint_R f(x,y)\,dA = \int_a^b\left(\int_{g(r)}^{h(r)} f(r\cos\theta, r\sin\theta)\,r\,d\theta\right)dr \quad \text{for Type II regions}$$

Application: • Area (Type I) $= \int_{\alpha}^{\beta} \left(\int_{g(\theta)}^{h(\theta)} r \, dr \right) d\theta$

• Area (Type II) $= \int_{a}^{b} \left(\int_{g(r)}^{h(r)} r \, d\theta \right) dr$.

$\boxed{f(x,y)=1}$ & use Thms.

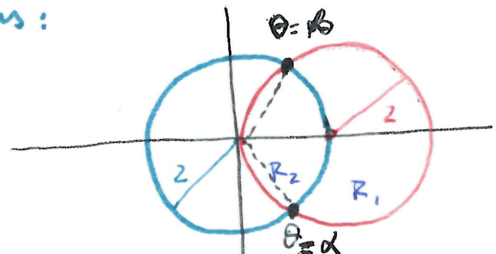
Example: Find the areas of (1) the region outside the circle $r=2$ (ie circle of radius 2 & center $=(0,0)$) & inside the circle $r = 4\cos\theta$ & (2) the region in between those.

Soln: Recall: $R = \{ r = 2a \sin\theta \}$ = circle of radius $|a|$ & center $(0,a)$

$R = \{ r = 2a\cos\theta \}$ = ——— $(a,0)$

(why? $(x-a)^2 + y^2 = a^2 \iff \underbrace{x^2+y^2}_{=r^2} + a^2 - 2ax = a^2$; $r^2 = 2ax = 2ar\cos\theta$; $\boxed{r = 2a\cos\theta}$)

(1) Draw the regions:



$C_1 = \{ r = 2 \}$

$C_2 = \{ 2 \cdot 2\cos\theta \}$ = radius 2 & center $(2,0)$.

(2) Find the intersections: $2 = r = 4\cos\theta \implies \frac{1}{2} = \cos\theta$; $\theta = \frac{\pi}{3}$ & $\theta = -\frac{\pi}{3}$ $\left(= 2\pi - \frac{\pi}{3} \right)$

R_1: Type I region: $\begin{cases} 2 \le r \le 4\cos\theta \\ -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \end{cases}$

$$A(R_1) = \iint_{R_1} 1 \, dA = \int_{-\pi/3}^{\pi/3} \left(\int_{2}^{4\cos\theta} r \, dr \right) d\theta = \int_{-\pi/3}^{\pi/3} \frac{r^2}{2} \Big|_{r=2}^{r=4\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3} 8\cos^2\theta - 2 \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 4(1+\cos 2\theta) - 2 \, d\theta = 2\theta + 2\sin 2\theta \Big|_{\theta=-\pi/3}^{\theta=\pi/3} = \boxed{4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\left[\begin{array}{l} \cos^2\theta - \sin^2\theta = \cos 2\theta \\ \cos^2\theta + \sin^2\theta = 1 \end{array} \right]$

R_2 = 

Area (R_2) = 2 (Area () + Area ())

first region = R_0; second area = $\frac{2^2 \cdot (\pi/3)}{2} = \frac{2\pi}{3}$

R_0 is Type II region $= \begin{cases} 0 \le r \le 2 \\ \frac{\pi}{3} \le \theta \le \arccos \frac{r}{4} \end{cases}$ but also Type I: $\begin{cases} \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 4\cos\theta \end{cases}$

$$\text{Area}(R_0) = \int_{\pi/3}^{\pi/2} \int_{0}^{4\cos\theta} r \, dr \, d\theta = \int_{\pi/3}^{\pi/2} 8\cos^2\theta \, d\theta$$

$$= (4\theta + 2\sin 2\theta) \Big|_{\theta=\pi/3}^{\theta=\pi/2} = \frac{4\pi}{6} + 2\left(0 - \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} - \sqrt{3} > 0 \checkmark$$

$$\implies \text{Area}(R_2) = 2\left(\frac{2\pi}{3} - \sqrt{3} + \frac{2\pi}{3} \right) = \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$