Lecture XIV : §14.4 Triple integrals

§1 Triple integrals in Rectangular Coordinates

$R = \{(x,y,z) \mid a_1 \le x \le b_1,\ a_2 \le y \le b_2,\ a_3 \le z \le a_3\} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is a rectangular region in $\mathbb{R}^3$.

GOAL: Given D in $\mathbb{R}^3$ a closed & bounded region and a continuous function $f : D \longrightarrow \mathbb{R}$, we want to compute the net volume of the solid in $\mathbb{R}^4$ bounded by D and the graph $w = f(x,y,z)$ of f in $\mathbb{R}^4$.

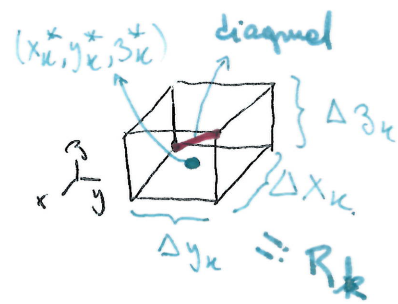
The volume is $\iiint_D f(x,y,z)\, dV$

- We can define the integral by Riemann sums: (eg D= cube, sphere in $\mathbb{R}^3$)

(1) Cover D by a collection of N boxes $R_k = [a_1^{(k)}, b_1^{(k)}] \times [a_2^{(k)}, b_2^{(k)}] \times [a_3^{(k)}, b_3^{(k)}]$ contained entirely in D, where
$$\begin{cases} b_1^{(k)} - a_1^{(k)} = \Delta x_k \\ b_2^{(k)} - a_2^{(k)} = \Delta y_k \\ b_3^{(k)} - a_3^{(k)} = \Delta z_k \end{cases}$$
& so $\boxed{Vol(R_k) = \Delta V_k = \Delta x_k \Delta y_k \Delta z_k}$

- (2) Pick (x_k^*, y_k^*, z_k^*) on R_k & form the "box in $\mathbb{R}^4$" with base R_k & height $f(x_k^*, y_k^*, z_k^*)$

If $\Delta = \max_{1 \le k \le N} \{\sqrt{\Delta x_k^2 + \Delta y_k^2 + \Delta z_k^2}\}$ = length of the diagonal



Def
$$\iiint_D f(x,y,z)\, dV = \lim_{\Delta \to 0} \underbrace{\sum_{k=1}^{N} \underbrace{f(x_k^*, y_k^*, z_k^*)\, \Delta V_k}_{\text{vol of box in } \mathbb{R}^4}}_{\text{Riemann sum}}$$
is the triple integral of f over D

Note: Continuity of f ensures that the limit exists and it is independent of all choices of the partition of D into boxes & all choices of points (x_k^*, y_k^*, z_k^*).

Remark: dV = element of volume in the integral becomes $dx\,dy\,dz$ (in some order)
$\rightsquigarrow$ Can calculate the integral by iteration (FUBINI-TYPE Result).

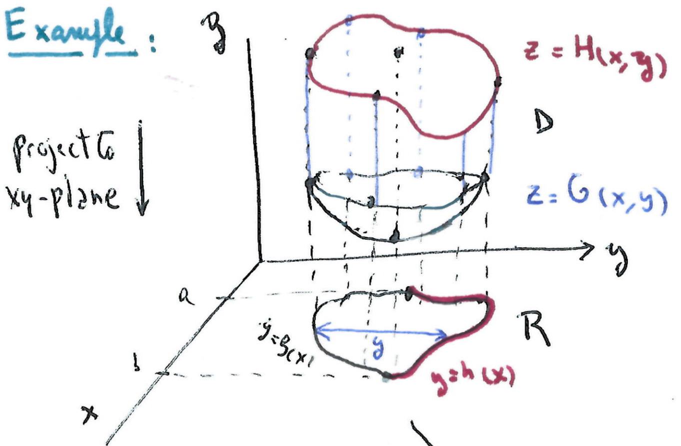
Now: we have 6 possible orders of integration!

§2 Finding limits of integration

6 orderings $\longleftrightarrow$ 6 Types of surfaces :

All Types: the dependency between variables (and their limits of integrations) grows from the outermost integral to the innermost one.

Example:



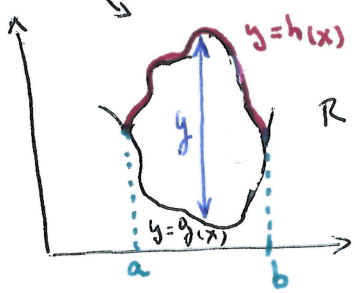
Suppose D in $\mathbb{R}^3$ is bounded above by the graph of a function $H(x,y)$ & below by the graph of $G(x,y)$

STEP 1: Identify $H_{(x,y)}$ & $G_{(x,y)}$

STEP 2: Project D down to the xy-plane to find the region R giving the boundaries in D as the graphs of $H_{(x,y)}$ & $G_{(x,y)}$ over the 2dim'l region R

STEP 3: R lies in $\mathbb{R}^2$ → Type I → Type II → decompose R into non-overlapping nice regions

In the example:



so type I $\quad g(x) \le y \le h(x)$

$a \le x \le b \qquad g(a) = h(a),\ g(b) = h(b)$

Need to find $g(x)$ & $h(x)$.

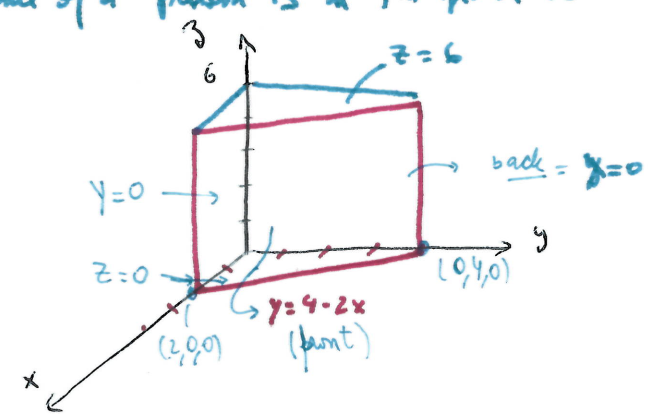
Theorem: If $D = \{(x,y,z)$ in $\mathbb{R}^3$: $a \le x \le b$, $g(x) \le y \le h(x)$, $G(x,y) \le z \le H(x,y)\}$

Then $$\iiint_D f(x,y,z)\,dV = \int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz \right) dy \right) dx.$$

By exchanging the roles of x, y & z we get 5 more results like this (6 descriptions of D & their corresponding different orders of integration).

Example: [Volume of a prism] Find the volume of a prism D in the first octant bounded by the planes $y = 4-2x$ & $z = 6$.

Sol:
- ALWAYS draw the solid first

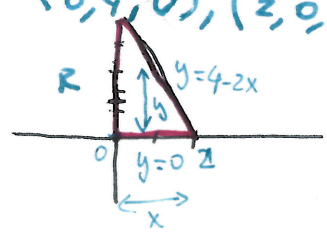


- $\text{Vol} = \iiint_D 1\,dV \rightsquigarrow$ need to find D.

Bounding planes : $\begin{cases} z=0 \\ z=6 \end{cases}$, $x=0$, $y=0$, $y=4-2x$

↳ draw the corresponding line in the xy-plane

(Intercepts = $(0,4,0), (2,0,0)$)



Many orders of integration :

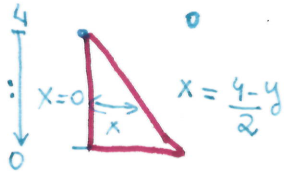
① $H(x,y)=6$, $G(x,y)=0$ defined on R =

$$\text{Vol} = \int_0^2 \left(\int_0^{4-2x} \left(\int_0^6 1 \, dz \right) dy \right) dx$$

So $D = \{(x,y,z) : 0 \le x \le 2, 0 \le y \le 4-2x, 0 \le z \le 6\}$

$$= \int_0^2 \int_0^{4-2x} 6 \, dy \, dx = \int_0^2 6(4-2x) \, dx = 24x - 6x^2 \Big|_0^2 = \boxed{24}$$

② R is also a Type II

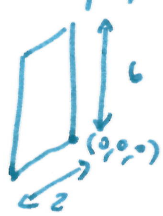


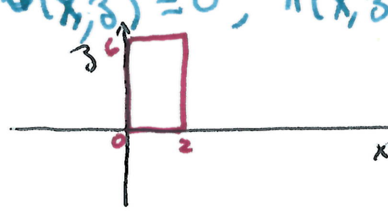
$D = \{(x,y,z) : 0 \le y \le 4, 0 \le x \le \frac{4-y}{2}, 0 \le z \le 6\}$

$$\text{So, Vol} = \int_0^4 \left(\int_0^{\frac{4-y}{2}} \left(\int_0^6 1 \, dz \right) dx \right) dy = \boxed{24}.$$

③ Could start by y in between the graphs of $G(x,z)=0$, $H(x,z)=4-2x$

Project to xz-plane : R =





$D = \{(x,y,z) : 0 \le z \le 6, 0 \le x \le 2, 0 \le y \le 4-2x\}$

$$\text{Vol} = \int_0^6 \left(\int_0^2 \left(\int_0^{4-2x} 1 \, dy \right) dx \right) dz = \int_0^2 \left(\int_0^6 \left(\int_0^{4-2x} 1 \, dy \right) dz \right) dx. \quad \text{(2 orders!)}$$

Note : The remaining 2 orders come from $D = \{(x,y,z) : 0 \le z \le 6, 0 \le y \le 4, 0 \le x \le \frac{4-y}{2}\}$

$R = \{(y,z) : 0 \le y \le 4, 0 \le z \le 6\}$ (project D to yz-plane)

§3 Changing the order of integration:

We can use this technique (showed in the example above) when integrating in one order is much more difficult than the other, for example when functions in one variable with no antiderivatives are involved.

Example: Compute $\displaystyle\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}} \, dy \, dx \, dz$

$D = \{(x,y,z) : 1 \le z \le 4, z \le x \le 4z, 0 \le y \le \pi^2\}$ & integral $= \iiint_D f(x,y,z) \, dV$

Q. Better order to integrate?

Can pick the order: $D = \{ (x,y,z) : 1 \le z \le 4,\ 0 \le y \le \pi^2,\ z \le x \le 4z \}$

Why? $\dfrac{\operatorname{sen}\sqrt{yz}}{x^{3/2}}$ has no antiderivative in y and z, but it has one in x, so x should be the innermost variable in the iterated triple integral.

So $$\iiint_D \frac{\operatorname{sen}\sqrt{yz}}{x^{3/2}}\,dV = \int_1^4 \left(\int_0^{\pi^2} \left(\int_z^{4z} \frac{\operatorname{sen}\sqrt{yz}}{x^{3/2}}\,dx \right) dy \right) dz = \int_1^4 \left(\int_0^{\pi^2} \operatorname{sen}\sqrt{yz}\,\frac{x^{-1/2}}{-1/2}\Bigg|_{x=z}^{x=4z} dy \right) dz$$

$$= \int_1^4 \left(\int_0^{\pi^2} \operatorname{sen}\sqrt{yz}\,(-2)\left(\frac{1}{\sqrt{4z}} - \frac{1}{\sqrt{z}} \right) dy \right) dz = \int_1^4 \left(\int_0^{\pi^2} \frac{\operatorname{sen}\sqrt{yz}}{\sqrt{z}}\,dy \right) dz$$

→ no antider. in y but one in z! (region: rectangle!)

$$= \int_0^{\pi^2} \left(\int_1^4 \frac{\operatorname{sen}\sqrt{yz}}{\sqrt{z}}\,dz \right) dy = \int_0^{\pi^2} -\cos\sqrt{yz}\,\frac{1}{\frac{1}{2}\sqrt{y}}\Bigg|_{z=1}^{z=4} dy$$

change order again!

$$= -2\int_0^{\pi^2} \frac{\cos\sqrt{4y}}{\sqrt{y}} - \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy = -2\int_0^{\pi^2} \frac{\cos 2\sqrt{y}}{\sqrt{y}} - \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy = -2\left(\frac{\operatorname{sen}\sqrt{y}}{2\cdot\frac{1}{2}} - \frac{\operatorname{sen}\sqrt{y}}{\frac{1}{2}} \right)\Bigg|_{y=0}^{y=\pi^2}$$

$$= -2\left(0-0-(0-0)\right) = \boxed{0}.$$