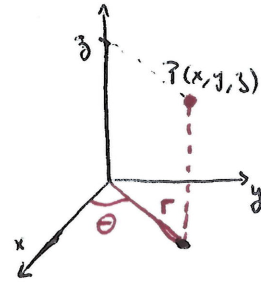
Lecture XV: § 14.5 Triple Integrals in Cylindrical & Spherical coordinates

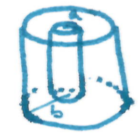
§1: Integration in Cylindrical coordinates

Recall: conversion rules

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right\} \xleftrightarrow[\text{cartesian}]{\text{polar to}} \left\{\begin{array}{l} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{array}\right.$$

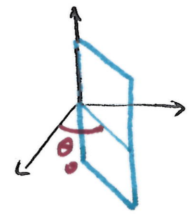


- Useful coordinate system for many examples:
  - Cylinders or cylinder shells $= \{(r,\theta,z) : 0 \leq a \leq r \leq b\}$

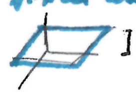


  - Cones: $\{(r,\theta,z) : z = ar\}$ $(a \neq 0)$
    (Eqn: $z^2 = a^2x^2 + a^2y^2$)

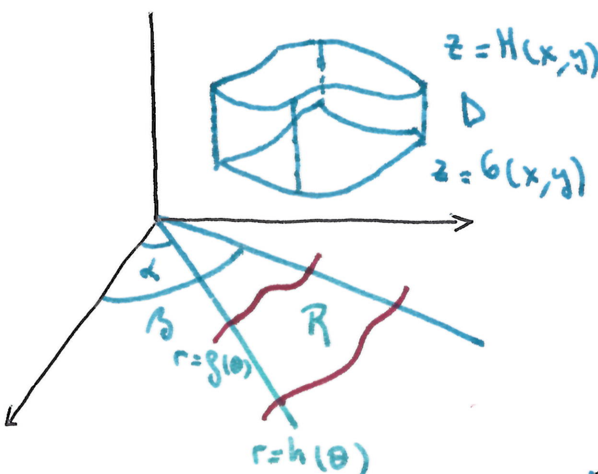
  - Vertical half-planes $= \{(r,\theta,z) : \theta = \theta_0\}$
    ↳ fixed angle



  - Horizontal planes: $\{(r,\theta,z) : z = a\}$

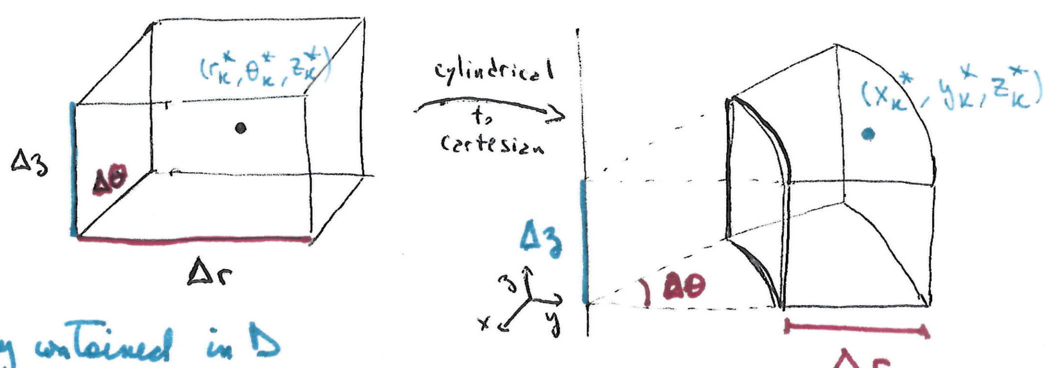


- How to integrate? Mimic the procedure to build double integrals with polar coordinates



Want to evaluate $\iiint_D f(x,y,z)\, dV$.

STEP 1: Partition D into N boxes (cylindrical wedges) formed by changes of Δr, $\Delta\theta$ & Δz in the coordinate directions.



- Keep only those boxes entirely contained in D
- Pick a point $(r_k^*, \theta_k^*, z_k^*)$ in Box_k, write its cartesian coordinates
  $(x_k^*, y_k^*, z_k^*) = (r_k^* \cos\theta_k^*, r_k^*\sin\theta_k^*, z_k^*)$

ΔV_k = Volume of the Box_k $= (r_k^* \Delta r\, \Delta\theta) \cdot \Delta z$
↳ Area of the base in polar coordinates (Lecture XIII)

STEP 2: Construct "Boxes" in 4-dimensions with base = Box k & height $f(x_k^*, y_k^*, z_k^*)$

The Rieman sum $= \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*) \, \mathrm{Vol}(\mathrm{Box}_k)$ approximates the triple integral

Write $\Delta = \max_{1\le k\le N} \{\Delta r, \Delta\theta, \Delta z\}$ (max of the dimensions of each box in D)

$$\iiint_D f(x,y,z)\, dV = \lim_{\Delta\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*) \Delta V_k \qquad \text{(Lecture XIV)}$$

Triple integral in cartesian coords.

$$= \lim_{\Delta\to 0} \sum_{k=1}^{N} f(r_k^*\cos\theta_k^*, r_k^*\sin\theta_k^*, z_k^*) \underbrace{r_k^*}_{} \Delta r\Delta\theta\Delta z$$

→ factor coming from cylindrical coords.

$$= \iiint_D f(r\cos\theta, r\sin\theta, z)\, r \, dV_{(r,\theta,z)}$$

(triple integral in cylindrical coordinates)

- Typical situation: compute integral by iterations (picture on page 1).
  - D lies between the graphs of 2 functions $H(x,y)$, $G(x,y)$
  - Project D to the xy-plane. Call R the planar region.
  - Assume R has a nice description in polar coordinates, e.g. $\begin{cases} \alpha \le \theta \le \beta \\ 0 \le g(\theta) \le r \le h(\theta) \end{cases}$ $(\beta-\alpha \le 2\pi)$ (Type I region in polar coords).

$dV = (dA)\, dz$

$$\iiint_D f(x,y,z)\, dV = \iint_R \underbrace{\int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz}_{} \, dA.$$

Once we compute the innermost integral, we can use polar coordinates to compute the double integral over R. Alternatively:

Theorem If $D = \{(r,\theta,z) : \alpha \le \theta \le \beta,\ 0 \le g(\theta) \le r \le h(\theta),\ G(r\cos\theta, r\sin\theta) \le z \le H(r\cos\theta, r\sin\theta)\}$, and f is continuous on D, then

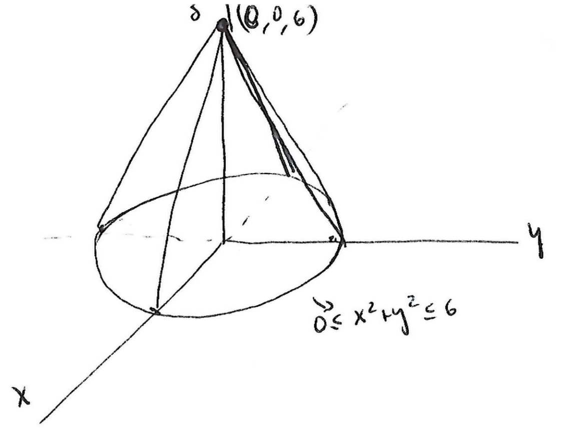
$$\iiint_D f(x,y,z)\, dV = \int_\alpha^\beta \left(\int_{g(\theta)}^{h(\theta)} \left(\int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, \underline{\underline{r}}\, dz \right) dr \right) d\theta.$$

($d\theta\, dr$ order of integration if R is a type II polar region).

Example: Mass of the solid cone $D = \{(x,y,z) : 0 \le z \le 6 - \sqrt{x^2+y^2},\ 0 \le x^2+y^2 \le 36\}$ with density $\rho(x,y,z) = 7 - z$.

Soln Draw the cone:

$\left[\begin{array}{l} (x,y)=(0,0) \Rightarrow 0 \le z \le 6 \\ x^2+y^2=36 \Rightarrow z=0 \end{array} \right.$ ⇒ upside down cone



Describe D in cylindrical coordinates.

$r^2 = x^2+y^2$

$D = \{(r,\theta,z): \ 0\le z\le 6-r, \ 0\le r\le 6\}$

R is a circle = rectangle in polar coords $=[0,6]\times[0,2\pi]$

Mass $= \iiint_D \rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^6\int_0^{6-r}(7-z)\,r\,dz\,dr\,d\theta$

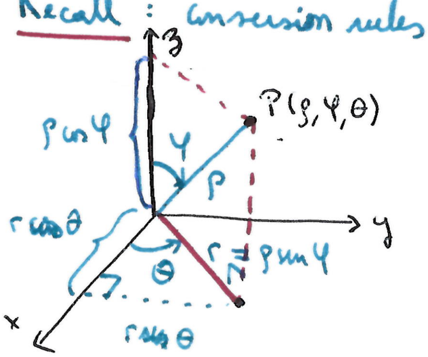
$$=\int_0^{2\pi}\int_0^6 r\left(7z-\frac{z^2}{2}\right)\Big|_{z=0}^{z=6-r} dr\,d\theta = 2\pi\int_0^6 r(6-r)\left(7-\frac{6-r}{2}\right)dr$$

$$=2\pi\int_0^6 (6r-r^2)\left(\underbrace{7-3}_{=4}+\frac{r}{2}\right)dr = 2\pi\int_0^6 24r - r^2 - \frac{r^3}{2}\,dr = 2\pi\left(12r^2-\frac{r^3}{3}-\frac{r^4}{8}\right)\Big|_{r=0}^{r=6}$$

$$=\boxed{396\pi}$$

§2 Integration in Spherical Coordinates

Recall: conversion rules

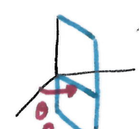


$$\begin{cases} x=(\rho\sin\varphi)\cos\theta \\ y=(\rho\sin\varphi)\sin\theta \\ z=\rho\cos\varphi \end{cases} \quad \xleftrightarrow[\text{cartesian}]{\text{spherical to}} \quad \begin{cases} \rho^2=x^2+y^2+z^2 \\ \tan\theta=\frac{y}{x} \\ \cos\varphi=\frac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

$0\le\varphi\le\pi$
$0\le\theta\le 2\pi$
$\rho\ge 0$

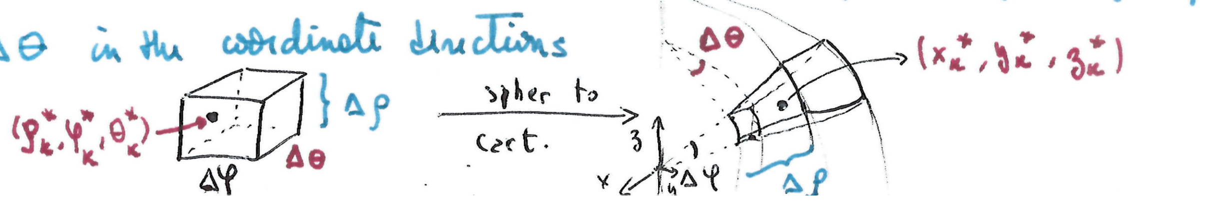
- Useful coordinate system for many examples:
  - Sphere w/center $(0,0,0)$ & radius a $=\{(\rho,\varphi,\theta): \rho=a\}$ 
  - Cone $=\{(\rho,\varphi,\theta): \varphi=\varphi_0\}$ $\quad\varphi_0\neq 0,\frac{\pi}{2},\pi$

  - Vertical half-plane: $\{(\rho,\varphi,\theta): \theta=\theta_0\}$



  - Horizontal plane $z=a$ $\to \{(\rho,\varphi,\theta): \rho=a\sec\varphi \quad 0\le\varphi<\pi/2\}$ $(a>0)$
    $\to \{(\rho,\varphi,\theta): \rho=a\sec\varphi \quad \frac{\pi}{2}<\varphi\le\pi\}$ $(a<0)$
  - Cylinder, radius $a>0$ $=\{(\rho,\varphi,\theta): \rho=a\frac{1}{\sin\varphi} \quad 0<\varphi<\pi\}$
- How to integrate? Similar to the cylindrical case....

STEP 1: Partition D in $\mathbb{R}^3$ into N boxes (spherical) formed by changes of $\Delta\rho$, $\Delta\varphi$ & $\Delta\theta$ in the coordinate directions

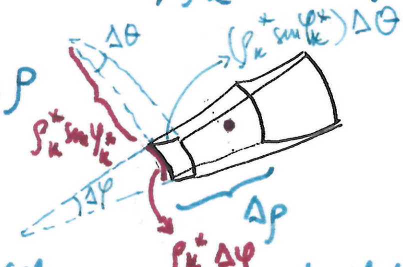


- Only keep those spherical boxes contained entirely in D
- Pick a point $(\rho_k^*, \varphi_k^*, \theta_k^*)$ on Box_k & write its cartesian coordinates

$$(x_k^*, y_k^*, z_k^*) = (\rho_k^* \sin\varphi_k^* \cos\theta_k^*, \ \rho_k^* \sin\varphi_k^* \sin\theta_k^*, \ \rho_k^* \cos\varphi_k^*)$$

$\Delta V_k =$ Volume of the Box_k in (x,y,z) coords. $\cong (\rho_k^* \Delta\varphi)(\rho_k^* \sin\varphi_k^* \Delta\theta)\,\Delta\rho$

$$= (\rho_k^*)^2 \sin\varphi_k^* \,\Delta\rho\,\Delta\varphi\,\Delta\theta.$$



STEP 2 : Construct "boxes" in 4 dimensions with base = spherical box_k & height $f(x_k^*, y_k^*, z_k^*)$

The Riemann sum $= \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*) \,Vol(Box_k)$ approximates $\iiint_D f(x,y,z)\,dV$

when $\Delta = \max_{1\le k\le N} \{\Delta\rho, \Delta\varphi, \Delta\theta\} \to 0$. (max of the dimensions of the spherical boxes covering D)

Conclusion: $$\iiint_D f(x,y,z)\,dV = \lim_{\Delta\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*)\,\Delta V_k$$

$$= \lim_{\Delta\to 0} \sum_{k=1}^{N} f(\rho_k^* \sin\varphi_k^* \cos\theta_k^*, \rho_k^* \sin\varphi_k^* \sin\theta_k^*, \rho_k^* \cos\varphi_k^*)\, {\rho_k^*}^2 \sin\varphi_k^* \,\Delta V_{(\rho,\varphi,\theta)}$$

$$= \iiint_D f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \underbrace{\rho^2\sin\varphi}_{\text{factor}}\ dV_{(\rho,\varphi,\theta)}$$

§3 Finding limits of integration   Depends on the example

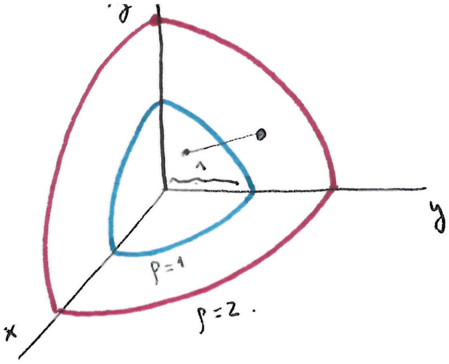
Theorem: Given $D = \{(\rho,\varphi,\theta) \quad 0 \le g(\varphi,\theta) \le \rho \le h(\varphi,\theta), \ a \le \varphi \le b, \ \alpha \le \theta \le \beta\}$ (radius in between the graphs of 2 functions) $[\,0 \le b-a \le \pi$ & $0 \le \beta-\alpha \le 2\pi\,]$

Then $$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \left(\int_a^b \left(\int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\ \rho^2\sin\varphi\ d\rho \right) d\varphi \right) d\theta$$

Note: The projection of D to the (φ,θ) was a rectangle, so we could have also chosen the order $d\rho\,d\theta\,d\varphi$.

Example: $D =$ region in the first octant between the spheres of radius 1 & 2 centered at $(0,0,0)$ & $f(x,y,z) = (x^2+y^2+z^2)^{-3/2}$.

Compute $\iiint_D f(x,y,z)\,dV$.



$D = \{(\rho,\varphi,\theta) : 1 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}\}$

→ spherical rectangle!

$f(x,y,z) = (\rho^2)^{-3/2} = \frac{1}{\rho^3}.$

$$\iiint_D f(x,y,z)\,dV = \int_0^{\pi/2}\left(\int_0^{\pi/2}\left(\int_1^2 \frac{1}{\rho^3}(\rho^2 \sin\varphi)\,d\rho\right)d\varphi\right)d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \frac{1}{\rho}\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\left(\ln|\rho|\,\Big|_{\rho=1}^{\rho=2}\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}(\ln 2)\sin\varphi\,d\varphi\,d\theta = \ln 2\int_0^{\pi/2} -\cos\varphi\,\Big|_{\varphi=0}^{\varphi=\pi/2} d\theta = \ln 2\int_0^{\pi/2} 1\,d\theta$$

$$= \boxed{\frac{\pi}{2}\ln 2}$$

• More examples in recitation / HW.