Lecture XVI: §14.6 Integrals for mass calculations

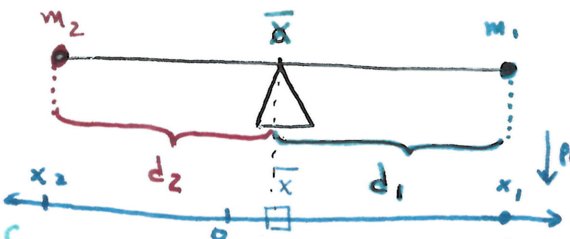
GOAL: Define and calculate the center of mass of various objects in 1-, 2- & 3-dim

Def: The center of mass of an object is the point where all of the mass of the object is concentrated. When the object is supported at its center of mass, there is no net torque acting on the body and it will remain in static equilibrium

Q: Given a solid object with irregular shape & variable density, where is the point at which all of the mass of the object would be located if it were treated as a point mass?

§1. One-dimensional objects (discrete case)

Find it with a "well-known playground principle": If 2 people with masses m_1 & m_2 sit at distance d_1 & d_2 from the pivot point of a massless seesaw, then the seesaw balances provided $\boxed{m_1 d_1 = m_2 d_2}$.



generalizations: ① Find the location $\bar{x}$ of the pivot point if we know the location & mass of the 2 objects. Call them x_1, m_1 & x_2, m_2, resp.

Since $d_1 = x_1 - \bar{x}$, $d_2 = -(x_2 - \bar{x})$ & the seesaw is at equilibrium, then:

$m_1(x_1 - \bar{x}) = m_2(-x_2 + \bar{x})$, equiv. $\boxed{\bar{x} = \dfrac{m_1}{m_1+m_2}\, x_1 + \dfrac{m_2}{m_1+m_2}\, x_2}$

(convex combination of x_1, x_2)

- $\bar{x}$ is the balance point or center of mass of the 2-mass system.

② Several points on a line. If we have n objects, each located at position x_i and mass m_i $(i = 1, \ldots, n)$, the balanced condition becomes:

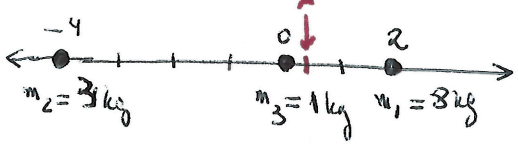
$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = 0$$

Solution: The center of mass $\bar{x}$ of the n-mass system is

$$\bar{x} = \frac{m_1}{m_1+\cdots+m_n}\, x_1 + \frac{m_2}{m_1+\cdots+m_n}\, x_2 + \cdots + \frac{m_n}{m_1+\cdots+m_n}\, x_n = \boxed{\frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}}$$

Convex combination of $x_1, x_2, \ldots, x_n$ (scalars are ≥ 0 & sum to 1)

Example Find the point at which the system balances:



$$\bar{x} = \frac{8\cdot 2 + 4(-3) + 1\cdot 0}{8+3+1} = \frac{4}{12} = \frac{1}{3}$$

(Total mass = 12)

§2 Continuous objects in one-dimension:

Setting: thin rod or wire with density $\rho(x)$ (ie it varies along the length of the body). The density has units of mass per length.

GOAL: Determine the location of the balance point $\bar{x}$.

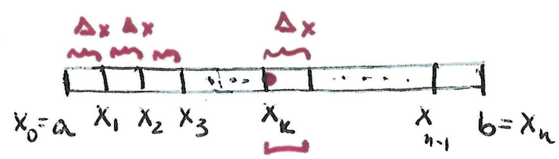
Method: Use the slice-and-sum strategy (in short, discretize the problem!)

STEP 1: The rod is the interval $[a,b] \to \{x : a \le x \le b\}$. We divide it into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. The grid points are

$$x_0 = a,\ x_1 = a + \Delta x,\ \dots,\ x_k = a + k\Delta x,\ \dots,\ x_n = b.$$

STEP 2: Assign to each subinterval the mass $= \rho(x_k)\,\Delta x$ (density times the length of the interval).

$\Rightarrow$ We have discretize the problem!



↳ mass $= \rho(x_k)\cdot\Delta x = m_k$

Center of mass is w/ n subintervals
$$\bar{x}_n = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k} = \frac{\sum_{k=1}^{n} (\rho(x_k)\,\Delta x)\, x_k}{\sum_{k=1}^{n} (\rho(x_k)\,\Delta x)}$$

We get the center of mass in the continuous setting by taking the limit as $\Delta x \to 0$, equivalently $n \to \infty$. Look at the limits of both numerator & denominator expressions for $\bar{x}_n$.

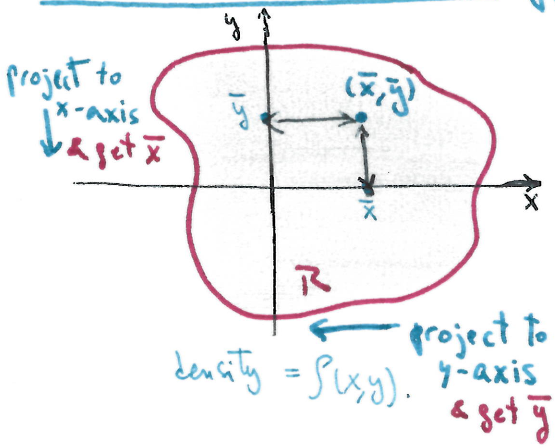
$$\text{num} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} (\rho(x_k)\, x_k)\,\Delta x = \int_a^b \rho(x)\, x\, dx$$ (use definition of $\int$ via Riemann Sums!)

$$\text{denom} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \rho(x_k)\,\Delta x = \int_a^b \rho(x)\, dx = \text{Total mass of the rod.}$$

[numerator is called the total momentum: weigh density by the distance to the origin!]

Theorem: Assume the density function is integrable on the interval $[a,b]$. The center of mass is located at the point $\bar{x} = \dfrac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$ $\begin{array}{l}\to \text{Total momentum}\\ \to \text{Total mass}\end{array}$

§ 3: Two-dimensional objects



- The center of mass (or balance point) has 2 coordinates $(\bar{x},\bar{y})$. If R has holes, this point could be outside R (e.g. DVD)
- We have a density function $\rho(x,y)$ which we assume is integrable on R, so we can compute the Total mass of R as $\iint_R \rho(x,y)\,dA$

- If we place a pivot at $(\bar{x},\bar{y})$, the object will be balanced.
- Instead of the distance to $(0,0)$, for $\bar{x}$ we need to weight $\rho(x,y)$ by the distance to the y-axis (this is the distance from the projection of the point (x,y) in the x-axis to the origin in the x-axis). This distance is x.

Theorem: Assume R is a closed bounded region in $\mathbb{R}^2$ and the density function is integrable on R. The coordinates of the center of mass of the object represented by R are:

$$\bar{x} = \frac{1}{m}\underbrace{\iint_R x\,\rho(x,y)\,dA}_{=\,M_y \text{ moment with respect to y-axis}} \quad \& \quad \bar{y} = \frac{1}{m}\underbrace{\iint_R y\,\rho(x,y)\,dA}_{=\,M_x \text{ moment w.r.t. x-axis}}.$$

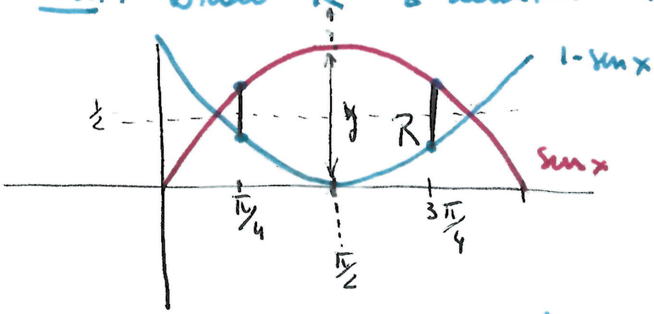
and $m = \iint_R \rho(x,y)\,dA$ is the mass.

Note: If the density function is constant, the center of mass depends ONLY on the shape of R. It's called centroid (& to compute $\bar{x}$ & $\bar{y}$ we can assume $\rho_{(x,y)} \equiv 1$)

Remark: Often, the geometry of R (eg symmetries) can help compute centroids (if ρ is constant). If ρ varies, we'll also need the function ρ to be symmetric wrt $(\bar{x},\bar{y})$.

Example: Let R be the region bounded by $y = \sin x$ & $y = 1 - \sin x$, where $\frac{\pi}{4} \le x \le 3\frac{\pi}{4}$. Compute the center of mass when (1) $\rho(x,y)$ is constant (2) $\rho(x,y) = x$

Soln: Draw R & determine the region type:



$\left(\frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} > 1 - \frac{\sqrt{2}}{2} \right)$

R is type I region.

(1) Region is symmetric about $x = \frac{\pi}{2}$, so $\bar{x} = \frac{\pi}{2}$. To compute $\bar{y}$, we take $\rho \equiv 1$

$$\bar{y} = \frac{\iint_R y \, dA}{\iint_R 1 \, dA}$$

$$\iint_R y \, dA = \int_{\pi/4}^{3\pi/4} \left(\int_{1-\sin x}^{\sin x} y \, dy \right) dx = \int_{\pi/4}^{3\pi/4} \frac{y^2}{2} \Big|_{y=1-\sin x}^{y=\sin x} dx$$

$$= \frac{1}{2}\int_{\pi/4}^{3\pi/4} \sin^2 x - (1-\sin x)^2 \, dx = \frac{1}{2}\int_{\pi/4}^{3\pi/4} 2\sin x - 1 \, dx$$

$$= \frac{1}{2}\left(-2\cos x - x \Big|_{x=\pi/4}^{x=3\pi/4} \right) = -\frac{1}{2}\left(\frac{4\sqrt{2}}{2} + \frac{3\pi}{4} - \frac{2\sqrt{2}}{2} - \frac{\pi}{4} \right)$$

$$= \boxed{\sqrt{2} - \frac{\pi}{4}} > 0 \ ✓$$

$$\therefore \iint_R 1 \, dA = \int_{\pi/4}^{3\pi/4} \left(\int_{1-\sin x}^{\sin x} 1 \, dy \right) dx = \int_{\pi/4}^{3\pi/4} (2\sin x - 1) \, dx = 2\left(\sqrt{2} - \frac{\pi}{4}\right), \text{ so } \boxed{\bar{y} = \frac{1}{2}}$$

(R is symmetric about $y = \frac{1}{2}$ so $\bar{y} = \frac{1}{2}$ ✓)

(2) We compute the total mass

• $\text{mass} = \iint_R x \, dA = \int_{\pi/4}^{3\pi/2} \left(\int_{1-\sin x}^{\sin x} x \, dy \right) dx = \int_{\pi/4}^{3\pi/4} x\,(2\sin x - 1) \, dx = \int_{\pi/4} 2x\sin x - x \, dx$

But $(x(-\cos x))' = -\cos x + x\sin x$.

$$= 2\left(\int_{\pi/4}^{3\pi/4} (x(-\cos x))' + \cos x \, dx \right) - \int_{\pi/4}^{3\pi/4} x \, dx = 2\left(-x\cos x \Big|_{x=\pi/4}^{x=3\pi/4} + \sin x \Big|_{x=\pi/4}^{x=3\pi/4} . \right) - \frac{x^2}{2}\Big|_{x=\pi/4}^{x=3\pi/4} = \boxed{(4\sqrt{2} - \pi)\pi}$$

• $M_y = \iint_R y \cdot x \, dA = \int_{\pi/4}^{3\pi/4} x \left(\int_{1-\sin x}^{\sin x} y \, dy \right) dx = \int_{\pi/4}^{3\pi/4} \frac{2x\sin x - x}{2} \, dx = \frac{1}{2}\,\text{mass}.$

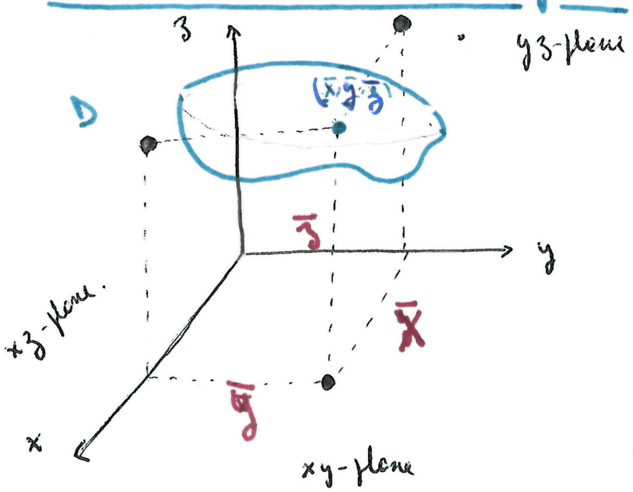
Conclusion $\bar{y} = \frac{1}{2}$

$$\bullet\, M_x = \iint_R x \cdot x \, dA = \int_{\pi/4}^{3\pi/4} \left(\int_{1-\sin x}^{\sin x} x^2 \, dy \right) dx = \int_{\pi/4}^{3\pi/4} x^2 (2\sin x - 1)\, dx$$

$$= 2\left(2x\sin x - (x^2-2)\cos x\right)\Big|_{\pi/4}^{3\pi/4} - \frac{x^3}{3}\Big|_{\pi/4}^{3\pi/4} = \sqrt{2}\,\pi\left(1+\frac{5}{16}\pi\right) - \frac{13\pi^3}{3\cdot 32}.$$

so $\bar{x} = \dfrac{M_x}{m} = \dfrac{\sqrt{2}\left(1+\frac{5}{16}\pi\right) - \frac{13\pi^3}{3\cdot 32}}{4\sqrt{2}-\pi} > \dfrac{\pi}{2}.$

§4: Three-dimensional objects



- The center of mass has 3 coordinates $(\bar{x}, \bar{y}, \bar{z})$
- We have a density function $\rho(x,y,z)$ which is integrable on D (closed & bounded region in $\mathbb{R}^3$) so the total mass is $\iiint_D \rho(x,y,z)\, dV$.

As with 2d, for $\bar{x}$, we need to weigh $\rho(x,y,z)$ by the distance to the yz-plane. This distance is x.

Theorem: Assume D is a closed bounded region in $\mathbb{R}^3$ & the density function is integrable on D. The coordinates of the center of mass of the region are:

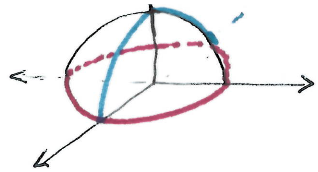
$$\bar{x} = \frac{1}{m} \underbrace{\iiint_D x\,\rho(x,y,z)\,dV}_{\substack{\shortparallel \\ M_{yz} \text{ moment w.r.t. the } yz\text{-plane}}} \;;\quad \bar{y} = \frac{1}{m} \underbrace{\iiint_D y\,\rho(x,y,z)\,dV}_{\substack{\shortparallel \\ M_{xz} \text{ moment wrt the } xz\text{-plane}}} \;;\quad \bar{z} = \frac{1}{m} \underbrace{\iiint z\,\rho(x,y,z)\,dV}_{\substack{\shortparallel \\ M_{xy} \text{ moment wrt the } xy\text{-plane}}}$$

where $m = \iiint_D \rho(x,y,z)\, dV$ is the total mass of D.

Note: • As with 2D, if the density is constant, the center of mass is independent of ρ & symmetry can help determine its coordinates. For concrete calculations we can take $\rho \equiv 1$.

Example: Find the center of mass of the interior of a hemisphere of radius 1 with its base on the xy-plane if (1) $\rho(x,y,z)$ is constant
(2) $\rho(x,y,z) = 2 - \sqrt{x^2+y^2+z^2}$.

Soln (1) The region D is



It's symmetric wrt the xz- & yz-planes, so $\bar{x} = \bar{y} = 0$

Thus, the center of mass lies on the (positive) z-axis.

We assume $\rho = 1$ so $m = \frac{1}{2}$ Vol (unit sphere) $= \iiint_D dV$ we compute it in spherical coordinates $D = \{(\rho, \varphi, \theta) : 0 \le r \le 1 \quad 0 \le \theta \le 2\pi, \ 0 \le \varphi \le \frac{\pi}{2}\}$

- $m = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^2 \sin\varphi \, dr\, d\varphi\, d\theta = 2\pi \int_0 \sin\varphi \left.\frac{r^3}{3}\right|_{r=0}^{r=1} d\varphi$

$$= \frac{2\pi}{3}(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi/2} = \frac{2\pi}{3}.$$

- $\bar{z} = \dfrac{M_{xy}}{m}$ $\qquad M_{xy} = \iiint_D z\, dV \underset{\text{use spherical coords.}}{=} \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^3 \sin\varphi\cos\varphi \, dr\, d\varphi\, d\theta =$

$$= 2\pi \int_0^{\pi/2} \cos\varphi \sin\varphi \left.\frac{r^4}{4}\right|_{r=0}^{r=1} d\varphi = \frac{\pi}{4}\int_0^{\pi/2} \sin(2\varphi)\, d\varphi$$

$$= \frac{\pi}{4}\left(\frac{-\cos 2\varphi}{2}\right)\Big|_{\varphi=0}^{\varphi=\pi/2} = \frac{\pi}{8}\left(-(-1-1)\right) = \frac{\pi}{4}$$

so $\bar{z} = \dfrac{\pi/4}{2\pi/3} = \boxed{\dfrac{3}{8}}$

(2) In this case, ρ & D are both symmetric with respect to the xz- & yz-planes so $\bar{x} = \bar{y} = 0$. The center of mass lies in the (positive) z-axis.

- $m = \iiint_D 2 - \sqrt{x^2+y^2+z^2}\, dV \underset{\text{spherical coords}}{=} \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (2-r)(r^2\sin\varphi)\, dr\, d\varphi\, d\theta$

$$= 2\pi \int_0^{\pi/2}\int_0^1 2r^2\sin\varphi - r^3\sin\varphi \, dr\, d\varphi = 2\pi \int_0^{\pi/2} \left.\frac{2r^3}{3}\sin\varphi - \frac{r^4}{4}\sin\varphi\right|_{r=0}^{r=1} d\varphi$$

$$= 2\pi \int_0^{\pi/2}\left(\frac{2}{3} - \frac{1}{4}\right)\sin\varphi\, d\varphi = 2\pi \cdot \frac{5}{12}\left(-\cos\varphi\Big|_{\varphi=0}^{\varphi=\pi/2}\right) = \boxed{\frac{5\pi}{6}}$$

- $M_{xy} = \iiint_D (2 - \sqrt{x^2+y^2+z^2})\, z\, dV \underset{\text{sph coords}}{=} \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 ((2-r)\, r\cos\varphi)\, r^2\sin\varphi\, dr\, d\varphi\, d\theta$

$$= 2\pi\left(\int_0^{\pi/2}\frac{1}{2}\sin\varphi\cos\varphi\, d\varphi - \int_0^{\pi/2}\frac{1}{5}\cos\varphi\sin\varphi\, d\varphi\right) = \frac{3\pi}{10}\int_0^{\pi/2}\sin 2\varphi\, d\varphi = \boxed{\frac{3\pi}{10}}$$

So $\boxed{\bar{z} = 9/25}$