Lecture XVII: §14.7 Change of variables in multiple integrals

GOAL: Extend substitution methods (CHANGE OF VARIABLES) for integrals over $\mathbb{R}$ to integrals over $\mathbb{R}^2$ or $\mathbb{R}^3$

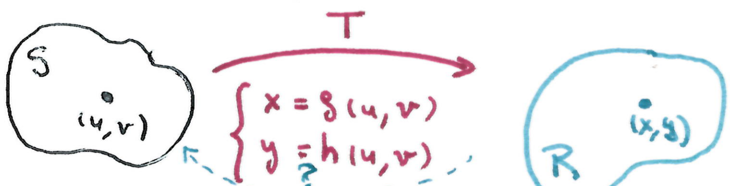
(Examples: double integrals in POLAR coordinates
triple integrals in CYLINDRICAL & SPHERICAL coordinates)

In these examples we had to (1) add a factor (r for polar / cylindrical, $\rho^2 \sin\varphi$ for spherical) to the function we are integrating & (2) determine new limits of integration.

§1 Transformations in the plane:

Def: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(u,v) \longmapsto (g(u,v), h(u,v))$ write $x = g(u,v)$, $y = h(u,v)$.

If we restrict T to a region S in $\mathbb{R}^2$, then T maps the region S "point by point" to a region $R = T(S)$ in $\mathbb{R}^2$ (R is the image of S under the transformation T)



- **Ideally**: The transform T should be invertible (at least in the interior of R), meaning we can find a map $W : R \longrightarrow S$ such that

invertible = 1-to-1, ie $T(u_1,v_1) = T(u_2,v_2)$ only when $u_1 = u_2$ & $v_1 = v_2$

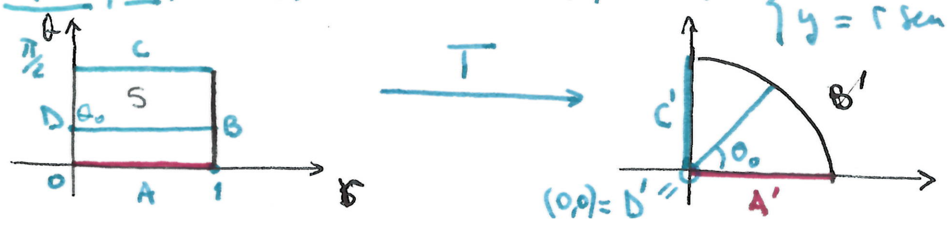
$(x,y) \longmapsto (p(x,y), q(x,y))$

$$\begin{cases} \bullet\ W\circ T(u,v) = W(\underbrace{g(u,v)}_{x}, \underbrace{h(u,v)}_{y}) = (p(g(u,v),h(u,v)), q(g(u,v),h(u,v))) = (u,v) \quad (\text{for } (u,v) \text{ in } S) \\ \bullet\ T\circ W(x,y) = T(\underbrace{p(x,y)}_{u}, \underbrace{q(x,y)}_{v}) = (g(p(x,y),q(x,y)), h(p(x,y),q(x,y))) = (x,y) \quad (\text{for } (x,y) \text{ in } R) \end{cases}$$

If that's the case, we say W is the inverse function of T and write $W = T^{-1}$.

- We can weaken the condition to T being invertible on the interior of R.
$T : \text{int}(S) \longrightarrow \text{int}(R)$ & $W = T^{-1} : \text{int}(R) \longrightarrow \text{int}(S)$.

Example: Polar to cartesian: $T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $S = \underbrace{[0,1]}_{r} \times \underbrace{[0,\frac{\pi}{2}]}_{\theta}$



METHOD
- Map the 4 edges in the boundary of S
write $A' = T(A)$, etc.
- inside of S ↦ inside or outside?

In this case T is not 1-To-1 because the edge $D=\{(0,\theta): 0\leq\theta\leq\frac{\pi}{2}\}$ maps entirely to the point $(0,0)$. However, it's 1-To-1 when $r\neq 0$ (so in particular, in $R\setminus\{(0,0)\}$, which contains the interior of R)

R = quarter of the unit circle

In this case: $W: R\setminus\{(0,0)\} \longrightarrow S\setminus D = (0,1]\times[0,\frac{\pi}{2}]$

$$(x,y) \longmapsto \begin{cases} \left(\sqrt{x^2+y^2},\ \tan^{-1}\left(\frac{y}{x}\right)\right) & \text{if } x\neq 0 \\ \left(\sqrt{x^2+y^2},\ \frac{\pi}{2}\right) & \text{if } x=0. \end{cases}$$

Remark:

- Since we will be using T (& W) To simplify integration, we will ALWAYS want T to be continuous & all the partials g_u, g_v, h_u, h_v to be continuous. (the same will be true for W).

① Factors to add = Jacobian determinants (for double integrals)

Def: Given a transformation $T: S \longrightarrow R$ $\begin{cases} x=g(u,v), \\ y=h(u,v) \end{cases}$ where g & h are differentiable on S (on int(S)), the Jacobian (determinant) of T is $J: S \longrightarrow \mathbb{R}$.

$$J_{(u,v)} := \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = (g_u h_v - g_v h_u)_{(u,v)}$$

Theorem: Let $T: S \longrightarrow R$ be a transformation $\begin{cases} x=g(u,v) \\ y=h(u,v) \end{cases}$ satisfying

(1) S is closed & bounded in $\mathbb{R}^2$

(2) $R = T(S)$ (image of S under T)

(3) T is 1-To-1 on the interior of S

(4) g, h have continuous first partial derivatives on int(S).

If f is continuous, then:

$$\iint_R f_{(x,y)}\, dA_{(x,y)} = \iint_S f_{(g(u,v),\, h(u,v))} \boxed{|J_{(u,v)}|}\, dA_{(u,v)}$$

← FACTOR TO ADD!

where $|J_{(u,v)}|$ is the absolute value of $J_{(u,v)}$.

Example above: $J_{(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$ so

$x = g(r,\theta) = r\cos\theta$

$y = h(r,\theta) = r\sin\theta$

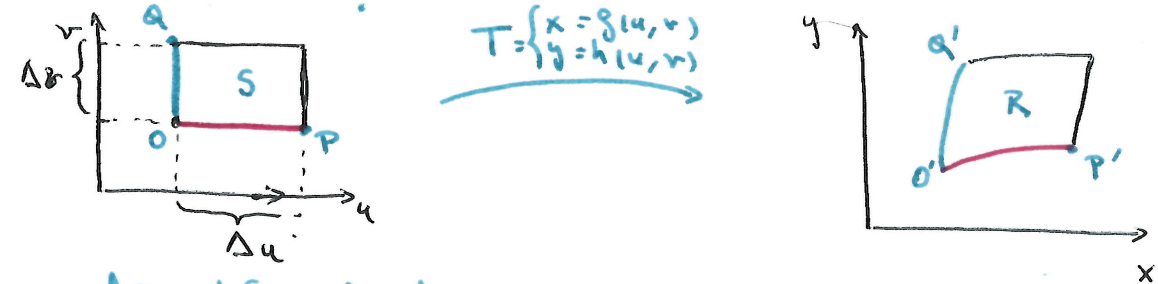
[S in $r\theta$-plane, R in xy-plane]

$|J_{(u,v)}| = |r| = r$

because we take $r\geq 0$ (it's the radius!)

Proof (idea): The Jacobian is a magnification (or reduction) factor that relates the area of a small region near a point O (in the uv-plane) to the area of the image of that region near $O' = T(O)$ (in the xy-plane).

These two areas are precisely the ones used in the Riemann sums needed to integrate $f(x,y)$ & $f(x(u,v), y(u,v))$.



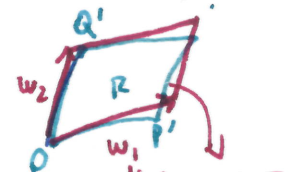
Write $O' = T(O)$
$P' = T(P)$
$Q' = T(Q)$
To simplify notation, assume $O = (0,0)$ in the uv-plane

Area of $S = \Delta u \, \Delta v$

Area of $R = ?$

We approximate R by a parallelogram, whose sides approximate the 2 "curved" segments $O'P'$ & $O'Q'$. Good approximations are linear, ie tangent lines!

We parameterize these 2 curved segments



approximate R by parallelogram

$$\begin{cases} O'P': \ \vec{r}_1(t) = \langle g(t,0), h(t,0) \rangle & 0 \le t \le \Delta u \\ O'Q': \ \vec{r}_2(s) = \langle g(0,s), h(0,s) \rangle & 0 \le s \le \Delta v \end{cases}$$

Tangent lines at O: $l_1(t) = \vec{r}_1(0) + \vec{r}_1{}'(0)\, t \qquad 0 \le t \le \Delta u$

$$= O' + \underbrace{\langle g_u(0,0), h_u(0,0) \rangle}_{\text{tangent direction!}} t$$

lin approx $\begin{cases} OP' \simeq \vec{r}_1{}'(0)\, \Delta x \\ OQ' \simeq \vec{r}_2{}'(0)\, \Delta y \end{cases}$

$$l_2(s) = \vec{r}_2(0) + \vec{r}_2{}'(0)\, s = O' + \underbrace{\langle g_v(0,0), h_v(0,0) \rangle}_{\text{tangent direction}} s \qquad 0 \le s \le \Delta v$$

So the parallelogram has edges $\vec{w}_1 = \langle g_u(0), h_u(0) \rangle \underset{\text{scalar!}}{\Delta u}$ & $\vec{w}_2 = \langle g_v(0), h_v(0) \rangle \underset{\text{scalar!}}{\Delta v}$

By Lecture 4, the area of the approximating parallelogram is

$$|\langle \vec{w}_1, 0 \rangle \times \langle \vec{w}_2, 0 \rangle| = \Delta u \, \Delta v \left| \begin{vmatrix} i & j & k \\ g_u & h_u & 0 \\ g_v & h_v & 0 \end{vmatrix} \right| = \Delta u \Delta v \underbrace{|g_u h_v - h_u g_v|}_{\text{absolute value!}}$$

so $\text{Area}(R) \approx |J_{(u,v)}| \, \Delta u \, \Delta v$.

Using Riemann sums:

$$\iint_R f(x,y)\, dA_{(x,y)} = \lim_{\Delta \to 0} \sum_{k=1}^{N} f(\underset{g(u_k^*,v_k^*)}{x_k^*}, \underset{h(u_k^*,v_k^*)}{y_k^*})\, J(u_k^*, v_k^*)\, \Delta u \Delta v = \iint_S f(g_{(u,v)}, h_{(u,v)})\, |J|\, dA_{(u,v)}$$

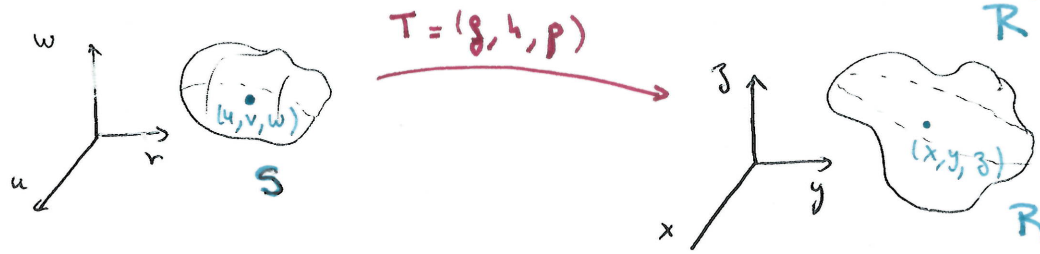
② How to find the limits of integration on the uv-plane?
Insert the map T (can do it in concrete examples)

§2. Transformations in space:

Def. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$(u,v,w) \longmapsto (x,y,z)$

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = p(u,v,w) \end{cases}$$

We restrict T to a closed & bounded region S & call $R = T(S)$ it's image under T.



Usual assumptions:
- T is 1-to-1 on the interior of S
- g, h, p are continuous & with first order partials also continuous.

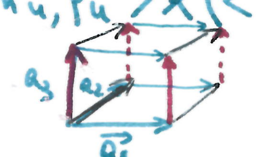
Def The Jacobian of T is $J : S \longrightarrow \mathbb{R}$

$$J(u,v,w) := \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{array}{c|ccc|c} & \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} & \\ & g_u & g_v & g_w & \leftarrow g \\ & h_u & h_v & h_w & \leftarrow h \\ & p_u & p_v & p_w & \leftarrow p \end{array}$$

(3×3 determinant)

absolute value of $J(u,v,w)$

Note: $|J(u,v,w)| = |(\langle g_u, h_u, p_u\rangle \times (\langle g_v, h_v, p_v\rangle)) \cdot \langle g_w, h_w, p_w\rangle$
is the volume of the parallelopiple



$\vec{a}_1 = \langle g_u, h_u, p_u\rangle$
$\vec{a}_2 = \langle g_v, h_v, p_v\rangle$
$\vec{a}_3 = \langle g_w, h_w, p_w\rangle$

Theorem: Let $T : S \longrightarrow R$ be a transformation $x = g(u,v,w)$, $y = h(u,v,w)$, $z = p(u,v,w)$ in $\mathbb{R}^3$ satisfying:
(1) S is closed & bounded in $\mathbb{R}^3$
(2) $R = T(S)$ (image of S under T)
(3) T is 1-to-1 on the interior of S
(4) g, h, p are continuous & have continuous first partial derivatives on int(S)

If f is continuous, then:

FACTOR TO ADD!

$$\iiint_R f(x,y,z)\, dV_{(x,y,z)} = \iiint_S f(g(u,v,w), h(u,v,w), p(u,v,w)) \boxed{|J(u,v,w)|}\, dV_{(u,v,w)}$$

Recall: 3×3 determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$\to$ 2×2 determinants

Examples: ① Cylindrical to cartesian
$S \qquad R$

$$\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \\ z = z \end{cases} \rightsquigarrow J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\operatorname{sen}^2\theta = \boxed{r}.$$

② Spherical to Cartesian
$S \qquad R$

$$\begin{cases} x = \rho \operatorname{sen}\varphi\cos\theta \\ y = \rho\operatorname{sen}\varphi\operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}$$

$$J(\rho,\varphi,\theta) = \begin{vmatrix} \operatorname{sen}\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\operatorname{sen}\varphi\cos\theta \\ \operatorname{sen}\varphi\operatorname{sen}\theta & \rho\cos\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \\ \cos\varphi & -\rho\operatorname{sen}\varphi & 0 \end{vmatrix} = \operatorname{sen}\varphi\cos\theta\,\rho^2\operatorname{sen}^2\varphi\cos\theta - \rho\cos\varphi\cos\theta(-\rho\operatorname{sen}\varphi\cos\varphi\cos\theta) - \rho\operatorname{sen}\varphi\operatorname{sen}\theta(-\rho\operatorname{sen}\theta\operatorname{sen}^2\varphi - \rho\cos^2\theta\operatorname{sen}\theta$$

$$= \boxed{\rho^2\operatorname{sen}\varphi}$$

§3 Strategies:

- INPUT: (1) Region R closed & bounded in xy-plane (resp, xyz-space)
  (2) f continuous on R $\quad f: R \longrightarrow \mathbb{R}$
- GOAL: Find S closed & bounded region in uv-plane (resp, uvw-space) & an appropriate continuous transformation $T: S \to R$ where
  - $T(S) = R$, T is invertible (1-to-1 in $\operatorname{int}(S)$)
  - components of T have continuous first partial derivatives on $\operatorname{int}(S)$

  that helps compute $\iint_R f(x,y)\,dA$ (resp. $\iiint_R f(x,y,z)\,dV$)

① Aim for simpler regions of integration S (eg S rectangular in polar, cylindrical or spherical coords)

② If choice of T: $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ is natural, getting $J(u,v)$ is immediate but computing S is harder (need to invert T!).

③ The function $f(x,y)$ can suggest changes of coordinates.

Eg 1: $f(x,y) = (x-y)\sqrt{x-2y}$ $\rightsquigarrow$ Pick $\begin{cases} u = x-y & = f(x,y) \\ v = x-2y & = g(x,y) \end{cases}$ so we know $W = T^{-1}$. Getting S is immediate ($S = W(R)$ & we know the region R in xy-plane) but we need T to find $J(u,v)$.

$\Rightarrow$ get T by writing x & y in terms of u & v:

$$T: \begin{cases} x = 2u - v \\ y = u - v \end{cases} \Longrightarrow J(u,v) = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = \boxed{1}.$$

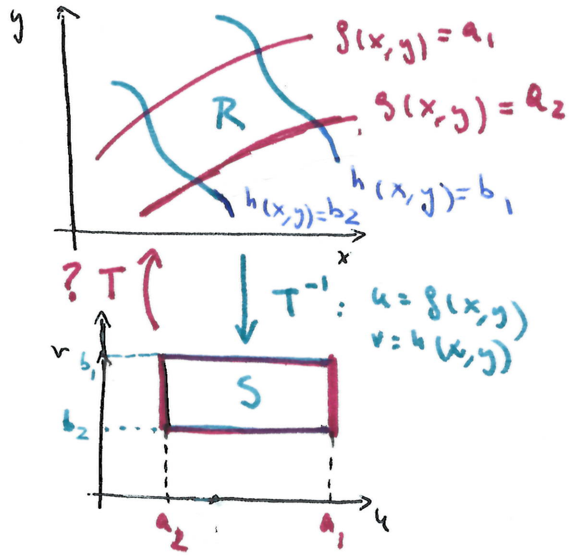
Eg 2: $f_{(x,y)} = (x+4y)^{3/2} \rightsquigarrow \begin{cases} u = x+4y \\ v = y \end{cases} \rightarrow \begin{cases} x = u-4v \\ y = v \end{cases}$

get S easily

$J_{(u,v)} = \begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix} = 1.$

④ The boundary of R can suggest changes of variables.

Typical example: R is bounded by 2 pairs of parallel curves

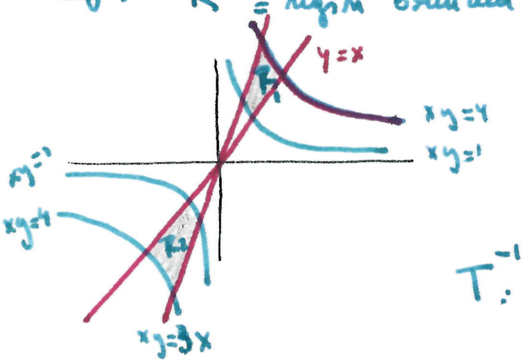


Write $\begin{cases} u = g(x,y) \\ v = h(x,y) \end{cases}$

Then $S = \{(u,v) : \begin{matrix} a_2 \le u \le a_1 \\ b_2 \le v \le b_1 \end{matrix}\}$ is a rectangle!

Write $\begin{matrix} x = x(u,v) \\ y = y(u,v) \end{matrix}$ to find T & get $J_{(u,v)}$

Eg: R = region bounded by the hyperbolas $xy=1$ & $xy=4$, & the lines $\frac{y}{x}=1$, $\frac{y}{x}=3$



$R = R_1 \cup R_2$. $\rightsquigarrow \iint_R f_{(x,y)}\,dA = \iint_{R_1} f_{(x,y)}\,dA + \iint_{R_2} f_{(x,y)}\,dA$

$T^{-1}: \begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$

$S_1 = T^{-1}(R_1) = \{(u,v) : \begin{matrix} 1 \le u \le 4 \\ 1 \le v \le 3 \end{matrix}\}$

$S_2 = T^{-1}(R_2) = S_1$ Call it $S = [1,4]\times[1,3]$

Note: $\begin{matrix} uv = y^2 \\ \frac{u}{v} = x^2 \end{matrix}$ $\rightsquigarrow$ signs determine if (x,y) belong to $\overset{(+,+)}{R_1}$ or $\overset{(-,-)}{R_2}$.

$T_1 : S \longrightarrow R_1 \quad (u,v) \longmapsto \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right)$

$T_2 : S \longrightarrow R_2 \quad (u,v) \longmapsto \left(-\sqrt{\frac{u}{v}}, -\sqrt{uv}\right)$

T_1 & T_2 are 1-to-1 & partial derivatives are continuous on S:

$$J_1(u,v) = \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{v}}\frac{1}{\sqrt{u}}, & -\frac{1}{2}\frac{\sqrt{u}}{(\sqrt{v})^3} \\ \frac{1}{2}\frac{\sqrt{v}}{\sqrt{u}}, & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4}\frac{1}{v} = \frac{1}{2v} = J_2(u,v)$$

If $f(x,y) = e^{xy} \Rightarrow f(T(u,v)) = e^u$ & $\iint_R f_{(x,y)}\,dA = 2\iint_S e^u \left|\frac{1}{2v}\right| dA = 2\int_{v=1}^{3}\int_1^4 \frac{e^u}{2v}\,du\,dv$

$= \frac{2}{2}\int_1^3 \frac{e^u}{v}\Big|_{u=4}^{u=1} dv = (e-e^4)\ln|v|\Big|_{v=1}^{v=3} = \boxed{\frac{(e-e^4)\ln 3}{2}}$