Lecture XXVIII : §15.1 Vector fields

Eg: ① model velocity vectors of a fluid (eg air, water) moving in space. (at every pt we have a vector) [Hurricane & wind patterns; gravitational, magnetic & electric force fields]

② Gradient flow: of a function f. at (x,y) we have $\nabla f(x,y)$: vector in $\mathbb{R}^2$.

• (x,y,z) " " $\nabla f(x,y,z)$: vector in $\mathbb{R}^3$.

(see Recitation 7)

§1 Vector fields in 2 dimensions:

Definition: given 2 functions $f,g : R \longrightarrow \mathbb{R}$. A vector field in a region R in $\mathbb{R}^2$ is a function F that assigns to each point (x,y) in R a vector $\langle f(x,y), g(x,y)\rangle$

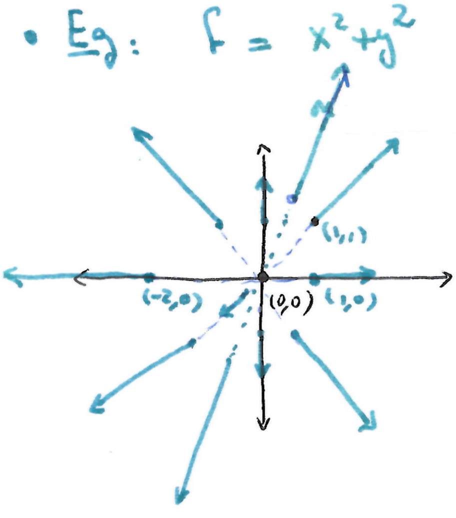
$\langle f,g\rangle = F : R \longrightarrow \mathbb{R}$ $\qquad F(x,y) = \langle f(x,y), g(x,y)\rangle$

• Note: F is continuous (resp. differentiable) on R if f & g are both continuous (resp. differentiable). In short, properties must hold componentwise

$$\lim_{(x,y)\to(a,b)} \underbrace{\langle f(x,y), g(x,y)\rangle}_{\text{vector in } \mathbb{R}^2} = \text{vector in } \mathbb{R}^2 = \langle \lim_{(x,y)\to(a,b)} f(x,y), \lim_{(x,y)\to(a,b)} g(x,y)\rangle$$

• For plotting, we draw $F(x,y)$ as a position vector where the tail is (x,y) [like we did when plotting gradients along level curves]

• Eg: $f = x^2+y^2 \rightsquigarrow F_{(x,y)} = \nabla f_{(x,y)} = \langle 2x, 2y\rangle$ $\qquad R = \mathbb{R}^2$



• For every (x,y) except $(0,0)$, the vector $F(x,y)$ points in the direction of $\langle x,y\rangle$ & has magnitude $= 2|\langle x,y\rangle|$.

• F points directly outward from the origin & the length of $F(x,y)$ increases with the distance to the origin.

This is an example of a radial vector field

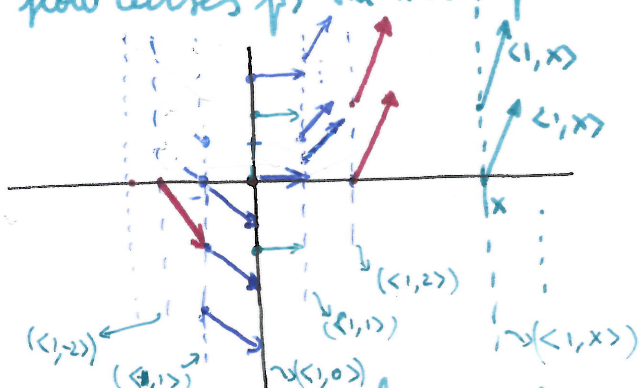
Note: Can place a particle at (x,y), and view $F(x,y)$ as the velocity of the particle. Then, can draw continuous curves that are aligned with the vector field (with it's sketch). These are called flow curves or streamlines.

These flow curves are everywhere tangent to F. In particular if the curve is $(x, y(x))$, then $\langle 1, y'(x) \rangle$ is parallel to $F(x,y) = \langle f(x,y), g(x,y) \rangle$

so $y'(x) = \dfrac{g(x,y)}{f(x,y)}$ & we can recover the function y from this.

Eg: Find and graph the flow curves for the vector field $\langle 1, x \rangle$:

Soln: ① Graph F:

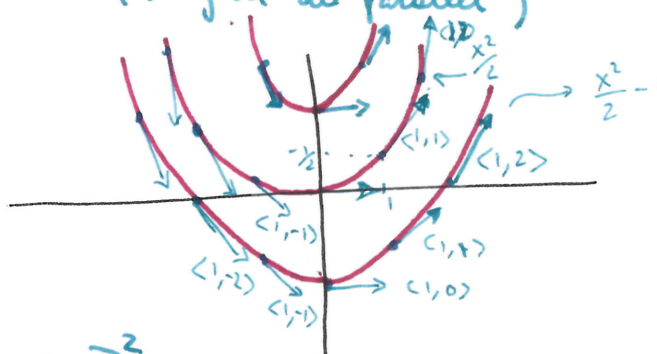


The vector field is indep. of y so along vertical lines, the vectors are ALL the same

② Find the flow curves. Assume they locally have the form $\langle x, y(x) \rangle$, so

$$y'(x) = \frac{x}{1} \rightsquigarrow y(x) = \int \frac{x}{1}\,dx = \frac{x^2}{2} + \text{constant}$$

$\Rightarrow$ get many curves (they are all parallel)



§2 Radial vector fields in $\mathbb{R}^2$:

Defining property: Their vectors point directly toward or away from $(0,0)$ at all points (except at $(0,0)$) & parallel to the position vector $\vec{r} = \langle x, y \rangle$.

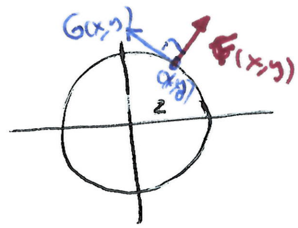
Examples: $F(x,y) = \dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{\langle x, y \rangle}{|\vec{r}|^p} = \underbrace{\dfrac{1}{|\vec{r}|^{p-1}}}_{\text{magnitude of } F(x,y)} \underbrace{\boxed{\dfrac{\vec{r}}{|\vec{r}|}}}_{\text{unit vector (direction)}}$ for some fixed p in $\mathbb{R}$.

Eg: Gravitational or electrostatic forces ($p=3$).

Definition: A vector field $F(x,y) = f(x,y)\langle x,y\rangle$ where $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$. is a radial vector field.

Eg. p-adic vector fields $F_{(p)}(x,y) = \dfrac{\langle x,y\rangle}{|\langle x,y\rangle|^{p}}$ $\quad (\langle x,y\rangle \neq \langle 0,0\rangle)$.

Example:



curve = circle of radius 2 (ρ) $\Rightarrow$ $\vec{P}(t) = \langle 2\cos t, 2\sin t\rangle$.

$\vec{P}'(t) = \langle -2\sin t, 2\cos t\rangle$

$\vec{P}''(t) = \langle -2\cos t, -2\sin t\rangle$

$\Downarrow$

Radial vector field $F_{(x,y)} = \dfrac{\langle x,y\rangle}{2}$

If $\langle x,y\rangle$ in circle $\Rightarrow$ $T(x,y) = \langle -y, x\rangle$ tangent vector field.

Rotational v. field $=$. $G(x,y) = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ unit tan. vector field.

In general: $F(x,y) = \dfrac{\langle x,y\rangle}{\sqrt{x^2+y^2}}$ is orthogonal to the line tangent to the circle C of radius ρ at (x,y).

$G(x,y) = \dfrac{\langle -y,x\rangle}{\sqrt{x^2+y^2}}$ rotational vector field is parallel to the line tangent to C at (x,y).

§3 Vector fields in $\mathbb{R}^3$:

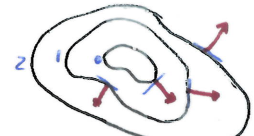
Definition. Given 3 functions $f, g, h : \mathbb{R}^3 \longrightarrow \mathbb{R}$, a vector field in $\mathbb{R}^3$ (or in a region D in $\mathbb{R}^3$) is a function F assigning $F(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$

Note F is continuous (resp. differentiable) if f, g, h are continuous (resp. differentiable)

- F is radial if $F(x,y,z) = \dfrac{\langle x,y,z\rangle}{|\langle x,y,z\rangle|^{p}}$ for some $p>0$.

Example $F = \nabla \varphi$ for some $\varphi : D \longrightarrow \mathbb{R}^3$ called a potential function.

§4. Gradient fields & potential functions



Note: $\varphi: R \longrightarrow \mathbb{R}$ differentiable function in a region R in $\mathbb{R}^2$, then the gradient field $F = \nabla\varphi = \langle \varphi_x, \varphi_y\rangle$ is orthogonal to the level curves of φ.

- Similarly, if $\varphi: D \longrightarrow \mathbb{R}$ differentiable in a region D in $\mathbb{R}^2$, then the gradient field $F = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$ is orthogonal to the level surfaces of φ
- Potential functions play the role of antiderivatives of vector fields. In particular, $\nabla(\varphi + c) = \nabla\varphi$.

§5 Equipotential curves and surfaces

$\varphi : R \longrightarrow \mathbb{R}$ $\quad R$ region in $\mathbb{R}^2$.

φ potential function for the vector field F in R.

Def: The level curves of φ are called equipotential curves (the potential is constant on these curves)

At every point (a,b) $\quad F_{(a,b)}$ is $\perp$ to the level curve $\varphi(x,y) = \varphi(a,b)$ at $(x,y) = (a,b)$

In this case: flow curves are $\perp$ to the level curves.