815.1 Vector fields

1. Model velocity vectors of a fluid (e.g., air, water) moving in space. (At every point, we have a vector.) [Hurricane wind patterns; gravitational, magnetic, electric force fields]

2. Gradient flow: at \((x, y)\), we have \(\nabla f(x, y)\): vector in \(\mathbb{R}^2\).

For function \(f\):

\(\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)\): vector in \(\mathbb{R}^2\).

(See Recitation 7)

81. Vector fields in 2 dimensions:

**Definition:** Given 2 functions \(f, g: \mathbb{R}^2 \to \mathbb{R}\), a vector field \(\mathbf{F}\) in \(\mathbb{R}^2\) is a function \(\mathbf{F}\) that assigns to each point \((x, y)\) in \(\mathbb{R}^2\) a vector \(\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle\).

\(\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle\)

**Note:** \(\mathbf{F}\) is continuous (resp., differentiable) in \(\mathbb{R}^2\) if \(f\) and \(g\) are both continuous (resp., differentiable). In short, properties must hold everywhere.

\[
\lim_{(x, y) \to (a, b)} \langle f(x, y), g(x, y) \rangle = \langle f(a, b), g(a, b) \rangle \quad \text{vector in } \mathbb{R}^2
\]

For plotting, we draw \(\mathbf{F}(x, y)\) as a position vector where the tail is \((x, y)\) [like we did when plotting gradients along level curves].

**Example:** \(f = x^2 + y^2\) \(\implies \mathbf{F} = \nabla f = \langle 2x, 2y \rangle\)

- For every \((x, y)\) except \((0, 0)\), the vector \(\mathbf{F}(x, y)\) points in the direction of \((x, y)\) and has magnitude \(= 2|<x, y>|\).
- \(\mathbf{F}\) points directly outward from the origin & the length of \(\mathbf{F}(x, y)\) increases with the distance to the origin.

This is an example of a radial vector field.
Note: Can place a particle at \((x,y)\), and view \(F(x,y)\) as the velocity of the particle. Then, can draw continuous curves that are aligned with the vector field (as with its sketch). These are called flow curves or streamlines.

These flow curves are everywhere tangent to \(F\). In particular, if the curve is \((x, y(x))\), then \(\langle x, y'\rangle\) is parallel to \(F(x, y) = \langle f(x, y), g(x, y)\rangle\), so \(y'(x) = \frac{g(x, y)}{f(x, y)}\) and we can recover the function \(y\) from this.

E.g.: Find and graph the flow curves for the vector field \(\langle 1, x \rangle\):

**Solu:**

1. **Graph \(F\):**

   - The vector field is indefinite of \(y\).
   - Allowing vertical lines, the vectors are all the same.

2. **Find the flow curves.** Assume they have the form \(\langle x, y(x) \rangle\), so locally

   \[
y'(x) = \frac{x}{1} \quad \Rightarrow \quad y(x) = \int \frac{x}{1} \, dx = \frac{x^2}{2} + \text{constant}
   \]

   \(\Rightarrow\) set many curves (they are all parallel)

**8.2 Radial vector fields in \(\mathbb{R}^2\):**

**Defining property:** Their vectors point directly toward or away from \((0,0)\) at all points (except at \((0,0)\)) and parallel to the position vector \(\vec{r} = \langle x, y \rangle\).

**Examples:** \(F(x, y) = \frac{\vec{r}}{||\vec{r}||^2} = \frac{\langle x, y \rangle}{||\vec{r}||^2} = \frac{1}{||\vec{r}||^2} \frac{\vec{r}}{||\vec{r}||}\), for some fixed \(\vec{r}\) in \(\mathbb{R}\).

E.g.: Gravitational or electrostatic forces \(f = 3\). Magnitude of \(F(x, y)\) = unit vector (direction)
**Definition:** A vector field \( \mathbf{F}(x,y) = f(x,y)\mathbf{e}_x + g(x,y)\mathbf{e}_y \) is a radial vector field.

**Example:**

\[ \mathbf{F}(x,y) = \frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{x^2 + y^2}} \]

In general, \( \mathbf{F}(x,y) = \frac{\mathbf{e}_y}{\sqrt{x^2 + y^2}} \) is orthogonal to the line tangent to the circle \( C \) of radius \( p \) at \((x,y)\).

\[ \mathbf{G}(x,y) = \frac{\mathbf{e}_x - \mathbf{e}_y}{2\sqrt{x^2 + y^2}} \]

**Rotational vector field** is parallel to the line tangent to \( C \) at \((x,y)\).

3.3 **Vector fields in \( \mathbb{R}^3 \):**

**Definition:** Given 3 functions \( f, g, h : \mathbb{R}^3 \to \mathbb{R} \), a vector field in \( \mathbb{R}^3 \) (or in a region \( D \) in \( \mathbb{R}^3 \)) is a function \( \mathbf{F} \) assigning \( \mathbf{F}(x,y,z) = f(x,y,z)\mathbf{e}_x + g(x,y,z)\mathbf{e}_y + h(x,y,z)\mathbf{e}_z \).

**Note:** \( \mathbf{F} \) is continuous (resp. differentiable) if \( f, g, h \) are continuous (resp. differentiable).

- **\( \mathbf{F} \) is radial** if \( \mathbf{F}(x,y,z) = \frac{\mathbf{e}_r}{r} \) to some \( r > 0 \).

**Example:** \( \mathbf{F} = \nabla \psi \) to some \( \psi : D \to \mathbb{R}^3 \) called a potential function.

4. **Gradient fields & potential functions**

Note: \( \psi : \mathbb{R} \to \mathbb{R} \) differentiable function in a region \( R \) in \( \mathbb{R}^2 \), then the gradient field \( \nabla \psi = \langle \psi_x, \psi_y \rangle \) is orthogonal to the level curves of \( \psi \).

Similarly, if \( \psi : D \to \mathbb{R} \) differentiable in a region \( D \) in \( \mathbb{R}^2 \), then the gradient field \( \nabla \psi = \langle \psi_x, \psi_y, \psi_z \rangle \) is orthogonal to the level surfaces of \( \psi \).

Potential functions play the role of antiderivatives of vector fields. In particular, \( \nabla (\psi + c) = \nabla \psi \).
§5 Equi-potential curves and surfaces

ψ : R → R²  R upim u R².

ψ potential function for the vector field F in R.

Def. The level curves of ψ are called equi-potential curves. (The potential is constant on these curves.)

At any point (x, y) F(x, y) is ⊥ to the level curve ψ(x, y) = ψ(x₀, y₀) at (x₀, y₀) = (x₀, y₀).

In this case, flow curves are ⊥ to the level curves.

flow curves of F = ∇ψ

level curves of ψ = equi-potential curves.