Lecture XXIX: § 15.2 Line Integrals

- This generalizes integrals of parametrizations of curves used, e.g., to calculate lengths of curves. Define arc length parametrization (Lecture IX).

8. Scalar line integrals in the plane

Input:Curve in the xy-plane \( \vec{r}(t) = \langle x(t), y(t) \rangle \): \([a,b] \rightarrow \mathbb{R}^2\)

- \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) where \( D \) contains the plane curve \( C \)

Goal: Find the area of the surface bounded by \( C \) & the graph of \( f \) restricted to \( C \)

Case 1: Assume \( C \) is a smooth curve of finite length

parameterized by arc length: \( s(t) = \int_a^t |\vec{r}'(u)| \, du \) with \( s \) the arc length param

Strategy: Use Riemann Sums!

Step 1: Partition \([a,b]\) into \( n \) intervals

of equal length \( \Delta t = \frac{b-a}{n} \)

Pick \( t_k^* \) in \([t_{k-1}, t_k]\) as \( n \)th interval, & the corresponding point \( \vec{r}(t_k^*) \) on the curve \( C \).

Notice: the curve \( C \) gets subdivided into \( n \) pieces, \( C_k : [t_{k-1}, t_k] \rightarrow \mathbb{R}^2 \) \( k = 1, \ldots, n \)

\( C_k \) has arc length param, so their lengths are all \( = \Delta t \)

Step 2

We approximate the length of \( f \) on \( C \) by \( n \) pieces: each with base \( C_k \) & height

The area under the curve \( f(x(t), y(t)) \) can be approximated by

\[
\sum_{k=1}^n \Delta t \sum_{k=1}^n \Delta t \sum_{k=1}^n \left[ f \left( x(t_k^*), y(t_k^*) \right) \right] \Delta t
\]

We can generalize the construction by letting the partition of \([a,b]\) have intervals of different lengths \( \Delta t_k = t_k - t_{k-1} \)

Write \( \Delta t = \max_{1 \leq k \leq n} \Delta t_k \)
Definition: A curve \( C: \mathbf{r}(t) = \langle x(t), y(t) \rangle: [a, b] \to \mathbb{R}^2 \) is parameterized by arc length if for each \( t \) pick \( f: D \to \mathbb{R} \) smooth when \( C \) belongs to the region.

The line integral of \( f \) over \( C \) is

\[
\int_C f(\mathbf{r}(s)) \, ds = \int_a^b f(x(t), y(t)) \, |\mathbf{r}'(t)| \, dt = \lim_{\Delta t \to 0} \sum_{k=1}^n f(\mathbf{r}(t_k)) \Delta t_k = \int_a^b f(\mathbf{r}(t)) \, |\mathbf{r}'(t)| \, dt
\]

provided the limit exists for all partitions of \( C \) and all choices of \( \mathbf{r}(t_k) \).

If the limit exists we say \( f \) is integrable on \( C \).

Example: \( f = 1 \Rightarrow \int_C 1 \, ds = \text{length}(C) \).

Can use this to find the average temperature along the edge of a plate:

\[
\text{Average value of } f \text{ along } C = \frac{1}{\text{Length}(C)} \int_C f(x(t), y(t)) \, dt
\]

Case 2: Arbitrary parameterization:

Recall: \( s(t) = \int_a^t \mathbf{r}'(u) \, du \) arc length parameter.

Then \( ds = s'(t) \, dt = |\mathbf{r}'(t)| \, dt \) (substitution).

We use change of coordinates: \( \mathbf{r}(s(t)) = \mathbf{r}(t) \Rightarrow \mathbf{r}'(s(t)) = \mathbf{r}'(t) \Rightarrow s(t) = \text{arc length param} \).

\[
\int_C f(x(s(t)), y(s(t))) \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt = \int_C f(x(t), y(t)) \, ds
\]

Theorem (Evaluating scalar line integrals in \( \mathbb{R}^2 \))

Let \( f: D \to \mathbb{R} \) continuous on a region \( D \) in \( \mathbb{R}^2 \) and \( C: \mathbf{r}(t): [a, b] \to \mathbb{R}^2 \) parameteric curve in \( D \). Then:

\[
\int_C f \, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| \, dt = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt
\]

speed of particle.

Example length (\( C \)) = \( \int_C 1 \, ds = \int_a^b \sqrt{\mathbf{r}'(t)^2} \, dt \) (Lecture 1x).

[Now in Recitation 10].

3.2. Line integrals in \( \mathbb{R}^3 \)

Some ideas (Riemann sums, parameterizations) lead to the following.

Then: \( C: D \to \mathbb{R}^3 \) continuous on a region \( D \) in \( \mathbb{R}^3 \) containing a curve \( C: \mathbf{r}(t): [a, b] \to \mathbb{R}^3 \).

Then:

\[
\int_C f \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt
\]
Typical: \( C \) is described geometrically (e.g., line between two points) so we need to find the parametrization \( \mathbf{T}^{(t)} \) to compute these integrals.

\section*{3. Line Integrals of Vector Fields}

- Differences w/ line integrals of \( \mathbb{R} \)-valued functions:
  - 0 curves have orientation: e.g., \( \mathbf{C} \)
  - \( \mathbf{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) vector field \( \mathbf{F} \) vs \( \mathbf{f}: \mathbb{R} \rightarrow \mathbb{R} \).

- Value of line integral depends on the orientation: \( \int_C \mathbf{f} \, ds = -\int_C \mathbf{f} \, ds \).

- Recall: \( \mathbf{T}^{(t)}: [a, b] \rightarrow \mathbb{R}^2 \) tangent vector to \( C = \mathbf{T}^{(t)} \) for \( t \in [a, b] \), \( \mathbf{T}^{(t)} \).

- \( \mathbf{T}^{(t)}: F(x, y) \) vector \( \mathbf{F} \) tangential component = \( \text{scal} \, \mathbf{F} = \mathbf{F} \cdot \mathbf{T} \). (Scalar comp of \( \mathbb{R} \) in the dir. of \( \mathbf{T} \)).

- \( \mathbf{F}: C \rightarrow \mathbb{R} \) \( \mathbf{F}(x, y) = F(x(t), y(t)) \cdot \mathbf{T}^{(t)} \). \( \mathbb{R} \)-valued function

\[ \int_C \mathbf{F} \cdot \mathbf{T} \, ds \]

- Question: What about general parametrization:

\[ ds = |r''(t)| \, dt \]

\[ \therefore \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{T}^{(t)} |r''(t)| \, dt \]

- Concisely: write \( \mathbf{F} = (f, g, h) \), \( \mathbf{r}^{(t)} = (x(t), y(t), z(t)) \)

\[ \mathbf{T}^{(t)} = \frac{(x'(t), y'(t), z'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}} \]

\[ \text{Then} \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{T}^{(t)} \, |r''(t)| \, dt = \int_C \left( f(x(t), y(t), z(t)) \cdot x'(t) + g(x(t), y(t), z(t)) \cdot y'(t) + h(x(t), y(t), z(t)) \cdot z'(t) \right) \, dt \]

\[ = \int_C \left( \frac{dx}{dt} + g \frac{dy}{dt} + h \frac{dz}{dt} \right) = \int_C \mathbf{F} \cdot d\mathbf{r} \]

Remark: \( \mathbf{r}^{(t)} \) has tangent \( -\mathbf{T}^{(t)} \) so \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds = -\int_C \mathbf{F} \cdot d\mathbf{r} \).
84. Application: Work integrals

Recall: Work done by a constant force

\[ F = F_0 \text{ N} \]

\[ F_{\text{unit vector}} = \frac{F}{|F|} \]

Total work = 1481 \text{ N}m \cos \theta.

Q: What if \( \mathbf{F} \) varies with the point \( a \), the object moves along a curve \( C \) (in the plane or in space).

Q.4: \[ W = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F} \cdot \mathbf{T}'(t) \, dt \]

work done by the continuous force \( \mathbf{F} \) in moving an object along \( C \) in the positive direction.

85. Circulation and flux of a vector field

1. Throughout we assume \( C \) is a closed, smooth curve in \( \mathbb{R}^3 \) (oriented!)

\[ \mathbf{F}(a) = \mathbf{F}(b) \quad \text{and} \quad \mathbf{T}'(t) \neq \mathbf{0} \]

Definition: \( \mathbf{F}: D \rightarrow \mathbb{R}^3 \) out. vector field (\( C \) in \( D \)). The circulation of \( \mathbf{F} \) on \( C \) is \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \).

Idea: \( \mathbf{F} \) measures the work done around \( C \) in the positive direction; how much does \( \mathbf{F} \) contribute.

Method: Find \( \mathbf{F} \) and \( C \) \( \Rightarrow \) compute the integral.

2. \( C \) smooth oriented curve in a region \( R \) in \( \mathbb{R}^2 \) with no self-crossings:

Definition: \( \mathbf{F}: R \rightarrow \mathbb{R}^2 \) out. vector field. The flux of \( \mathbf{F} \) across \( C \) is \( \int_C \mathbf{F} \cdot \mathbf{n} \, ds \), where \( \mathbf{n} \) is the outer normal to \( C \).

Out normal \( \mathbf{n} = \hat{T} \times \hat{N} = \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{array} \right| = T_y \hat{i} - T_x \hat{j} + \hat{k} = <T_y, -T_x, 1>

\[ \mathbf{F} = <f, g> \]

Thus flux of \( \mathbf{F} \) across \( C \) is \( \int_C \mathbf{F} \cdot \mathbf{n} \, ds \), where \( \mathbf{n} \) is the outer normal to \( C \).

\[ \mathbf{F}(x(t), y(t), z(t)) = <f'(t), g'(t), h'(t)> \]

\[ \mathbf{n}(t) = [a, b] \rightarrow C \quad \mathbf{n}(t) = <x(t), y(t), z(t)> \]

\[ \mathbf{F}(x(t), y(t), z(t)) = <f(x(t), y(t), z(t)), g(x(t), y(t), z(t)), h(x(t), y(t), z(t))> \]

\[ \mathbf{F}(0) \rightarrow \mathbb{R}^2 = <T_y, -T_x> \]

Q: How do I pick the right one?

Solve: Go to \( \mathbb{R}^3 \)!

\[ \mathbf{T} = \alpha \mathbf{T}_1 + \beta \mathbf{T}_2 \]

Where \( \alpha, \beta \) \in \mathbb{R} \]