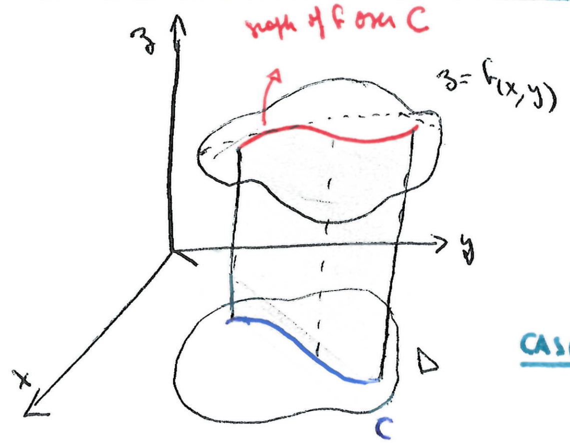
Lecture XXIX: §15.2 Line integrals

- This generalizes integrals of parametrizations of curves used, e.g., to calculate lengths of curves & define arc length parametrization (Lecture IX)

§1 Scalar line integrals in the plane



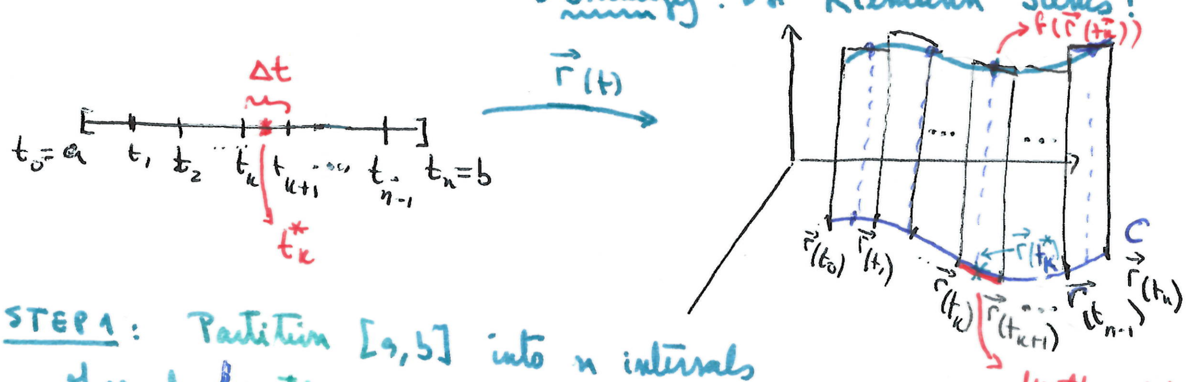
INPUT: Curve C in the xy-plane $\vec{r}(t) = \langle x(t), y(t)\rangle : [a,b] \longrightarrow \mathbb{R}^2$

- $f : D \longrightarrow \mathbb{R}$ where D contains the plane curve C

GOAL: Find the area of the surface bounded by C & the graph of f restricted to C

CASE 1: Assume C is a smooth curve of finite length parameterized by ARC LENGTH: $\underset{\text{ARC LENGTH}}{s(t) := \int_a^t |\vec{r}\,'(u)|\,du = t}$

- Strategy: Use Riemann Sums!



- STEP 1: Partition $[a,b]$ into n intervals of equal length $\Delta t = \dfrac{b-a}{n}$ $\rightsquigarrow$ points $t_k = a + k\Delta t$ $(0 \le k \le n)$

Pick t_k^* in $[t_{k-1}, t_k] = k^{th}$ interval, for each $k = 1,\ldots,n$ & the corresponding point $\vec{r}(t_k^*)$ on the curve C.

Notice: the curve C gets subdivided into n pieces C_k: $\vec{r} : [t_{k-1}, t_k] \longrightarrow \mathbb{R}^2$ $k=1,\ldots,n$

C has arc length param, so these lengths are all $= \Delta t$

- STEP 2

We approximate the graph of f over C by n pieces: each with base C_k & height $= f(\vec{r}(t_k^*))$

The Area under the curve $f(x(t), y(t))$ can be approximated by

$$\sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\,\Delta t$$

- We can generalise the construction by letting the partition of $[a,b]$ have intervals of different lengths $\Delta t_k = t_k - t_{k-1}$ $\quad a = t_0 \le t_1 < t_2 \cdots < t_{n-1} < t_n = b$

Write $\Delta t = \max\{\Delta t_k : 1 \le k \le n\}$

Definition: A curve $C: \vec{r}(t) = \langle x(t), y(t)\rangle : [a,b] \to \mathbb{R}^2$ is parameterized by arc length & pick $f: D \to \mathbb{R}$ smooth when C belongs to the region D.
The line integral of f over C is:

$$\int_C f(\vec{r}(s))\,ds = \int_C f(x(s), y(s))\,d\overset{\text{arc length param.}}{s} := \lim_{\Delta t \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\,\Delta t_k = \int_a^b f(\vec{r}(s))\,ds$$

provided the limit exists over all partitions of C & all choices of pts $(t_k^*)_{k=1}^n$.
If the limit exists we say f is integrable over C.

Example: $f = 1 \Rightarrow \int_C 1\,ds = \text{length}(C)$.

- Can use this to find the average temperature along the edge of a plate $:= \int_C T_{(x(s), y(s))}\,ds \Big/ \text{len}(C)$ — temp at pt $(x(s), y(s))$

In general: average value of a function f along C $= \dfrac{1}{\text{len}(C)} \int_C f(x(t), y(t))\,dt$.

CASE 2: Arbitrary parametrization:

Recall: $s(t) = \int_a^t |\vec{r}'(u)|\,du$ arc length parameter

Then $ds = s'(t)\,dt = |\vec{r}'(t)|\,dt$ (substitution)

We use change of coordinates: $\vec{r}_1(s(t)) = \vec{r}(t)$ & $\vec{r}(s) = \vec{r}_1(s(t))$ is arc length param.

$$\boxed{\int_C f(x(s), y(s))\,ds = \int_C f(\vec{r}_1(s))\,ds = \int_a^b f(\vec{r}(t))\underbrace{|\vec{r}'(t)|\,dt}_{=ds}}$$

Theorem (Evaluating scalar line integrals in $\mathbb{R}^2$.)
Let $f: D \to \mathbb{R}$ continuous on a region D in $\mathbb{R}^2$ & $C: \underset{\langle x(t), y(t)\rangle}{\vec{r}(t)}: [a,b] \to \mathbb{R}^2$ parametric curve in D. Then:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,\underset{\text{speed of particle.}}{|\vec{r}'(t)|}\,dt = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Example $\text{length}(C) = \int_C 1\,ds = \int_a^b |\vec{r}'(t)|\,dt$ (Lecture 1x).
[More in Recitation 10].

§2. Line integrals in $\mathbb{R}^3$

Same ideas (Riemann sums, parametrizations) lead to the following

Thm: $f: D \to \mathbb{R}$ continuous on a region D in $\mathbb{R}^3$ containing a curve: $C\ \underset{\langle x(t), y(t), z(t)\rangle}{\vec{r}(t)}: [a,b] \to \mathbb{R}^3$

Then:
$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|\,dt$$

Typical: C is described geometrically (e.g. line between 2 pts) so we need to find the parameterization $\vec{r}(t)$ to compute these integrals.

§3. Line integrals of Vector Fields:

- Differences w/ line integrals of $\mathbb{R}$-valued functions

$$\begin{cases} \text{① curves have orientation : eg } \bullet\!\!\longrightarrow\bullet \ , \ \circlearrowleft \\ \text{② } F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ vector field vs } f: \mathbb{R}^2 \longrightarrow \mathbb{R}. \end{cases}$$

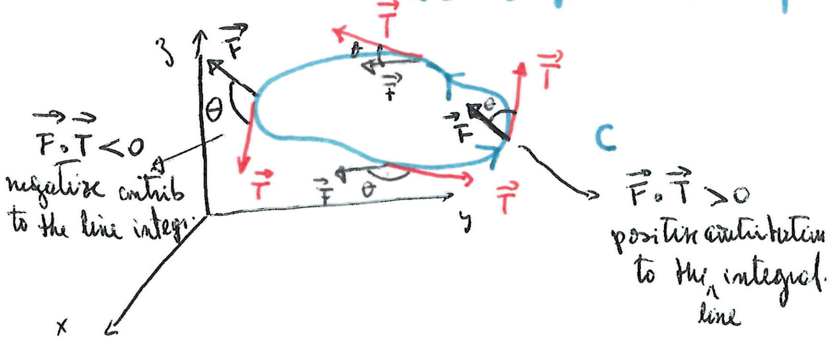
⇝ unique tangent vector (moves w/ the curve in the same direction)

- Value of line integrals depends on the orientation: $\int_C f\, ds = -\int_{C^{op}} f\, ds$.
- Recall: $\vec{T}: [a,b] \longrightarrow \mathbb{R}^2$ unit tangent vector to C $= \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

For t: $\vec{F}_{(x(t),y(t))}$ vector ⇝ tangential component $= \boxed{\operatorname{scal}_{\vec{T}} \vec{F} = \vec{F} \bullet \vec{T}}$. (scalar comp of $\vec{F}$ in the dir. of $\vec{T}$).

yields $f: C \longrightarrow \mathbb{R}$ $\quad f_{(x(t),y(t))} = F(x(t),y(t)) \bullet \vec{T}(t)$ $\quad$ $\mathbb{R}$-valued function

Def: The line integral of the vector field $\vec{F}$ over C is $\int_C \vec{F} \bullet \vec{T}\, ds$ where $\vec{T}$ is the unit tangent to C if C is parameterized by arc length.



Q: What about general parameterizations?

$ds = |\vec{r}'(t)|\, dt$ (arc length param.)

so $\displaystyle\int_C \vec{F} \bullet \vec{T}\, ds = \int_C \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)|\, dt$

$$= \boxed{\int_a^b \vec{F}_{(r(t))} \bullet \vec{r}'(t)\, dt.}$$

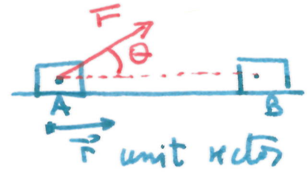
- Concretely: write $\vec{F}_{(x,y,z)} = \langle f, g, h \rangle_{(x,y,z)}$ & $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then
$$\int_C \vec{F} \bullet \vec{T}\, ds = \int_a^b F_{(\vec{r}(t))} \bullet \vec{r}'(t)\, dt = \int_a^b \{ f_{(x(t),y(t),z(t))} \cdot x'(t) + g_{(x(t),y(t),z(t))} \cdot y'(t)$$
$$+ h_{(x(t),y(t),z(t))} \cdot z'(t) \} dt$$
$$= \int_C f\,dx + g\,dy + h\,dz = \int_C F \bullet d\vec{r}.$$

$$\begin{cases} dx = x'(t)\,dt \\ dy = y'(t)\,dt \\ dz = z'(t)\,dt \end{cases}$$

Remark: C^{op} has tangent $= -\vec{T}_C$ so $\displaystyle\int_{C^{op}} \vec{F} \cdot \vec{T}_{C^{op}}\, ds = -\int_C \vec{F} \cdot \vec{T}_C\, ds$.

§4. Application: Work integrals



$F_{x\text{-comp}} = \vec{F}\cdot\vec{r} = |\vec{F}|\cos\theta$

<u>Recall</u>: Work done by a constant force

So, Total work $= |AB| \cdot |\vec{F}|\cos\theta$.

<u>Q</u>: What if $\vec{F}$ varies with the point & the object moves along a curve (in the plane or in space)?

<u>Def</u>: $$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$F: D \to \mathbb{R}^3$ cont (D in $\mathbb{R}^3$)
$C: \vec{r}(t): [a,b] \to \mathbb{R}^3$ curve in D.

<u>work</u> done by the continuous force $\vec{F}$ in moving an object along C in the positive direction.

§5. Circulation and flux of a vector field

$\boxed{\vec{F} = \text{velocity field for a moving fluid}}$

① . Throughout we assume C is a closed, smooth curve in $\mathbb{R}^3$ (oriented!)

closed: $\vec{r}(a) = \vec{r}(b)$; smooth: $\vec{r}\,'(t) \neq \vec{0}$

<u>Definition</u>: $\vec{F}: D \to \mathbb{R}^3$ cont. vector field (C in D). The <u>circulation</u> of $\vec{F}$ on C is $\int_C \vec{F}\cdot\vec{T}\,ds$.

<u>Idea</u>: It measures, as we travel along C in the positive direction how much does $\vec{F}$ contribute.

- same definition if C is a curve in $\mathbb{R}^2$ & $\vec{F}: D \to \mathbb{R}^2$ v. field in $\mathbb{R}^2$.

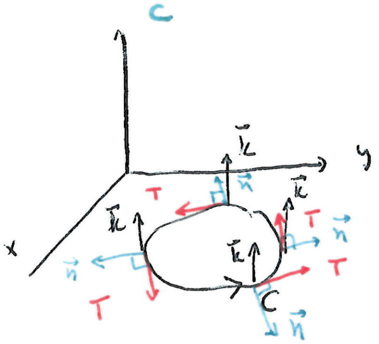
<u>Method</u>: Find $\vec{r}$ ⇝ find $\vec{T}$ ⇝ compute the integral.

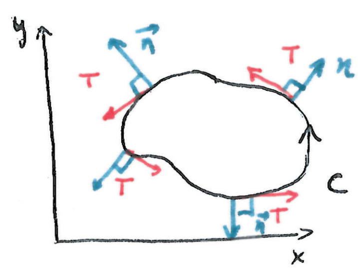
② . C smooth oriented curve in a region R in $\mathbb{R}^2$ with <u>NO</u> self crossings:

self crossing

<u>Definition</u>: $\vec{F}: R \to \mathbb{R}^2$ cont. vector field. The <u>flux</u> of $\vec{F}$ across C is

$\int_C \vec{F}\cdot\vec{n}\,ds$, where $\vec{n}$ is the <u>outer</u> <u>normal</u> to C.





2 normals ⇝ <u>Q</u>: How to pick the right one? ($\vec{n}$ & $-\vec{n}$)

<u>Soln</u>: Go to $\mathbb{R}^3$!

$\vec{k} = \langle 0,0,1\rangle$

$\vec{T} = \langle T_x, T_y, 0\rangle$

- Outer normal $\vec{n} = \vec{T}\times\vec{k} = \begin{vmatrix} i & j & k \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\,\vec{i} - T_x\,\vec{j} + 0\,\vec{k} = \langle T_y, -T_x, 0\rangle$

(right-hand rule!)

$\Rightarrow \vec{n}$ in $\mathbb{R}^2 = \boxed{\langle T_y, -T_x\rangle}$

- If $\vec{F} = \langle f, g\rangle$, then <u>flux</u> of $\vec{F}$ across C $= \boxed{\int_a^b (\tilde{f}(t)\,y'(t) - \tilde{g}(t)\,x'(t))\,dt}$
  - $F(x(t), y(t)) = \langle \tilde{f}(t), \tilde{g}(t)\rangle = \langle f(x(t), y(t)), g(x(t), y(t))\rangle$
  - $\vec{r}(t): [a,b] \to C \quad \vec{r}(t) = \langle x(t), y(t)\rangle$