Lecture XXX: §15.3 Conservative Vector fields

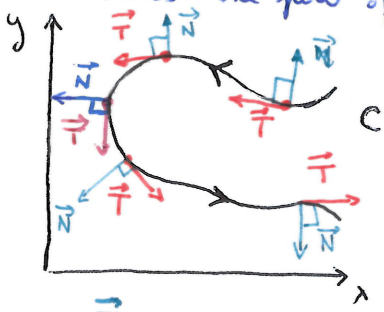
Last time: line integrals of scalar & vector fields along a curve C.

① **Circulation** $= \int_C \vec{F}\cdot\vec{T}\,ds$ $\quad \vec{T}_{(s)}$ = unit tangent vector | C closed smooth curve in $\mathbb{R}^3$

measures contribution of a v. field $\vec{F}$ (eg velocity field for a moving fluid) when a particle travels along C in the positive direction

② **Flux:** $= \int_C \vec{F}\cdot\vec{N}\,ds$ $\quad \vec{N}$ = unit vector normal to C ; C smooth curve with NO self-crossings

measures the flow of a force along a curve



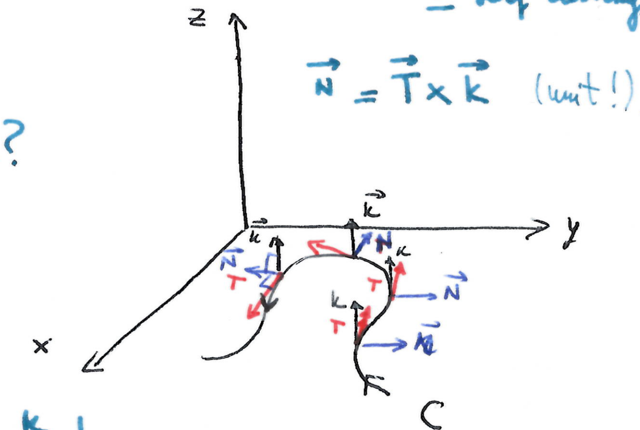
$\vec{T} = \langle T_x, T_y\rangle$

2 normal directions

Q: How to pick the right one?

Sln: Go to $\mathbb{R}^3$!

$\begin{cases}\vec{T} = \langle T_x, T_y, 0\rangle\\ \vec{k} = \langle 0,0,1\rangle\end{cases}$



- Outer normal (right-hand rule!): $\vec{N} = \vec{T}\times\vec{k} = \begin{vmatrix} i & j & k\\ T_x & T_y & 0\\ 0 & 0 & 1\end{vmatrix} = T_y\vec{i} - T_x\vec{j} + 0\vec{k} = \langle T_y, -T_x, 0\rangle$

$\Rightarrow \vec{N}$ in $\mathbb{R}^2$: $\vec{N} = \langle T_y, -T_x\rangle$ (unit vector & $\perp\vec{T}$).

- In coordinates: • $\vec{F}_{(x,y)} = \langle f(x,y), g(x,y)\rangle \rightsquigarrow F\circ\vec{r}(t) = \langle f(x(t),y(t)), g(x(t),y(t))\rangle = \langle\tilde{f}(t), \tilde{g}(t)\rangle$

• $\vec{r}(t) = \langle x(t), y(t)\rangle$

$$\text{Flux} = \int_C \vec{F}\cdot\vec{N}\,ds = \boxed{\int_a^b \left(\tilde{f}(t)\cdot y'(t) - \tilde{g}(t)\,x'(t)\right)dt}$$

§15.3 Conservation vector fields

GOAL: Answer 2 questions for a vector field $\vec{F}$.

① When is F the gradient of a potential function φ? If so, we say F is a CONSERVATIVE v. field.

② What are the properties of conservative vector fields? A: Path independence!

Ideally: Testing if F is conservative or not should produce the potential φ.

- Answer to ① depends on the properties of the region where $\vec{F}$ is defined $\begin{cases}\vec{F}: R \longrightarrow \mathbb{R}^2\\ \vec{F}: D \longrightarrow \mathbb{R}^3\end{cases}$ (R: region in $\mathbb{R}^2$, D: region in $\mathbb{R}^3$).

§.1 Types of curves and regions: $C: r(t) : [a,b] \longrightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$)

Def: • C is a simple curve if $\vec{r}(t_1) \neq \vec{r}(t_2)$ for all $a < t_1 < t_2 < b$.
that is, C never intersects itself between its endpoints. It doesn't have self-crossings.

• C is closed if $\vec{r}(a) = \vec{r}(b)$.

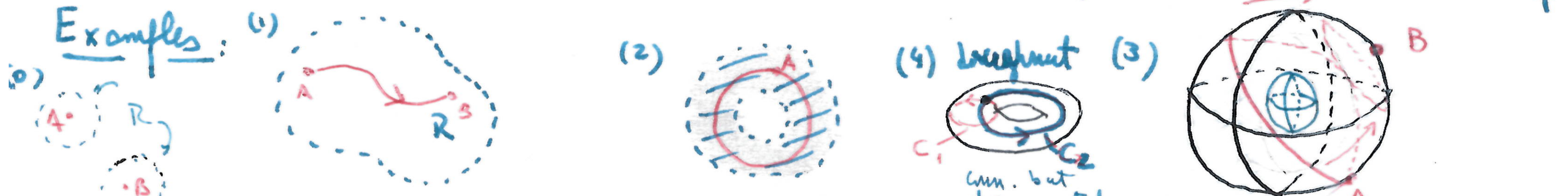
Examples.

simple & closed — simple & not closed — not simple & closed — not simple, not closed

Def: • An open region R of $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is connected if it is possible to connect any two points of R by a continuous curve lying in R [Other names: path connected]

• An open region R of $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is simply connected if it's connected & every simple closed curve in R can be deformed and contracted to a point in R.

Examples:



(0) not connected (2 connected components)

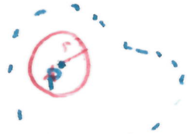
(1) connected & simply connected.

(2) connected but not simply connected

(4) conn. but not s. connected (can contract C_1 but NOT C_2)

(3) D in between 2 spheres connected & simply connected (can deform to a pole!)

Recall: open region



Given p in R we can find a radius $r > 0$ such that the ball $(p, r) = \{(x,y) : |(x,y) - p| < r\}$ lies in R.

§2 Test for conservative vector fields

Def: $\vec{F}$ v. field is conservative if there exists a scalar function $\varphi : R \to \mathbb{R}$ (or $\varphi: D \to \mathbb{R}$)
$F: R \longrightarrow \mathbb{R}^2$ (or $F: D \to \mathbb{R}^3$)
such that $\nabla \varphi = F$ on the region R (or D)

• Write $\vec{F} = \langle f, g, h \rangle$ & suppose $\nabla \varphi = \vec{F}$ is conservative w/ potential φ.

So $\varphi_x = f$, $\varphi_y = g$, $\varphi_z = h$.

Suppose $\vec{F}$ has continuous 1st partial derivatives:

$$\left.\begin{array}{l} f_y = (\varphi_x)_y = \varphi_{xy} \quad \text{cont} \\ g_x = (\varphi_y)_x = \varphi_{yx} \quad \text{cont} \end{array}\right\} \Longrightarrow f_y = \varphi_{xy} = \varphi_{yx} = g_x$$

mixed derivatives Thm.

Similarly: $f_z = \varphi_{xz}$ cont $\Rightarrow$ agree! ; $g_z = \varphi_{yz}$ cont $\Rightarrow$ agree!
$h_x = \varphi_{zx}$ cont $\qquad h_y = \varphi_{zy}$ cont

• Necessary conditions : $f_y = g_x$, $f_z = h_x$, & $g_z = h_y$. ($f_y = g_x$ for v.f. in $\mathbb{R}^2$)

Turns out they suffice if the region D (or R) is open, connected & simply connected

Theorem : Test for conservative vector fields :

Assume $F = \langle f, g, h \rangle$ is a vector field, $F: D \to \mathbb{R}^3$ where f, g, h have continuous first partials & D is an open, connected & simply connected region.

Then F is a conservative v. field in $D \iff$ (if and only if) $f_y = g_x$, $f_z = h_x$ & $g_z = h_y$

• If $F = \langle f, g \rangle$, is a vector field in $\mathbb{R}^2$ the condition is $f_y = g_x$ provided the same nice conditions on f & R hold.

Proof : See Lecture XXXI for ($\Leftarrow$) direction.

§3 Finding potential functions :

• $\boxed{\text{Methods = integration}}$ (find antiderivatives of f, g, h to get φ).

Eg: $F = \langle f, g \rangle$ where $f_y = g_x$ on $\mathbb{R}^2$.

($F = \langle e^x \cos y, -e^x \sin y \rangle$.)

① $\varphi_x = f \rightsquigarrow \varphi_{(x,y)} = \int f_{(x,y)}\, dx + C_{(y)}$

↳ constant in x : so it's a function in y.

② Then use $\varphi_y = g$ to guess what C is.

Eg: $\varphi_{(x,y)} = \int e^x \cos y \, dx + C_{(y)} = e^x \cos y + C_{(y)}$

$\Rightarrow \varphi_y = -e^x \sin y + C'_{(y)} = g(x,y) = -e^x \sin y$ so $C'_{(y)} = 0$

$\Rightarrow C_{(y)} = $ constant

so $\boxed{\varphi = e^x \cos y + \text{constant}}$

• For vector fields in $\mathbb{R}^3$, the method is the same, although ① $\varphi_x = f \rightsquigarrow \varphi_{(x,y,z)} = \int f_{(x,y,z)}\, dx + C_{(y,z)}$

constant depends only y & z.

② Use $\varphi_y = g$ to get an expression for $C_y(y,z)$, then

$C(y,z) = \int C_y(y,z)\, dy + \tilde{C}_{(z)} \Rightarrow \varphi = \int f_{(x,y,z)}\, dx + \int C_{y(y,z)}\, dy + \tilde{C}_{(z)}$

③ Use $\varphi_z = h$ to get $\tilde{C}'_{(z)}$ & then $\tilde{C}_{(z)} = \int \tilde{C}'_{(z)}\, dz + $ constant.

§4 Fundamental Theorem for line integrals & path independence

Theorem 1 Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ & φ a differentiable function. Then

$$\int_C \nabla\varphi \cdot d\vec{r} := \int_C \nabla\varphi \cdot \vec{T}\, ds = \varphi(B) - \varphi(A)$$

for all points A & B in R and all piecewise smooth oriented curves C in R from A to B.

- without loss of generality, we can assume C is smooth. Otherwise add subintegrals over smooth pieces ✓

Pf/ • Write $\vec{r}(t) : [a,b] \longrightarrow R$ the parametrization of C. $r(t) = \langle x(t), y(t), z(t)\rangle$

$A = \vec{r}(a)$, $B = \vec{r}(b)$.

Then $\vec{T}(t)$ = unit tangent $= \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \Rightarrow \displaystyle\int_C \nabla\varphi \cdot \vec{T}\, ds = \int_a^b \nabla\varphi(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$\int_C \nabla\varphi\cdot\vec{T}\, ds = \int_a^b \langle \varphi_x, \varphi_y, \varphi_z\rangle_{(x(t), y(t), z(t))} \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt$$

$$\underset{\boxed{\text{Chain Rule}}}{=} \int_a^b \frac{d\varphi}{dt}(x(t), y(t), z(t))\, dt \underset{\boxed{\text{Fund. Thm of Calculus}}}{=} \varphi(x(b), y(b), z(b)) - \varphi(x(a), y(a), z(a)) = \varphi(B) - \varphi(A)$$

□

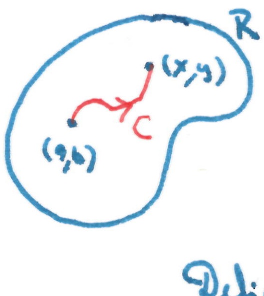
Def: Fix $\vec{F}$ a continuous vector field on R. If $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for all piecewise smooth curves C_1 & C_2 in R with the same initial & end points, then the line integral is *independent of the path*.

Claim: $\vec{F}$ conservative, then path independence holds. (by Theorem above)

Notably, if R is open connected region & $\vec{F}$ is cont., the converse holds!

Theorem 2 Let $\vec{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ or $\mathbb{R}^3$. If $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ is independent of the path, then $\vec{F}$ is conservative.

Proof: For simplicity, we prove the result for R in $\mathbb{R}^2$, so $\boxed{\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle}$



Fix (a,b) in R. Idea: Define φ as an "antiderivative along C"

For each (x,y) in R we can find a piecewise smooth path C in R joining (a,b) & (x,y) (in that direction).

Define $\boxed{\varphi(x,y) := \displaystyle\int_C \vec{F}\cdot d\vec{r}}$

GOAL: Show that $\nabla\varphi = F$

Wai

Note: path independence shows the definition of φ does not depend on the choice of path C.

Strategy: we will show that $\boxed{D_{\vec{u}}\varphi_{(x,y)} = \vec{F}_{(x,y)} \cdot \vec{u} \quad \text{for every unit vector } \vec{u} \text{ in } \mathbb{R}^2} \ (*)$

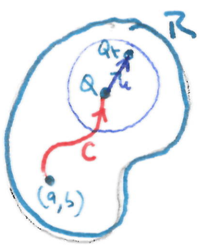
$\nabla\varphi \cdot \vec{u}$

Then, take $u = \langle 1,0 \rangle$ to get $\varphi_{x\,(x,y)} = f_{(x,y)}$; take $u = \langle 0,1 \rangle$ to get $\varphi_{y\,(x,y)} = g_{(x,y)}$ □

Proof of $(*)$: We compute $D_u \varphi_{(x,y)}$ using the definition. Write $Q = (x,y)$

R is open so we can find a ball $B((x,y),\delta)$ inside R for $\delta > 0$ small enough.

We can construct a piecewise smooth path C' from (a,b) to $Q_t = (x+tu_1, y+tu_2)$ by concatenating our path C with the line segment joining Q & Q_t



$$D_{\vec{u}}\varphi_{(x,y)} = \lim_{t\to 0} \frac{\varphi_{(x+tu_1,y+tu_2)} - \varphi_{(x,y)}}{t} \qquad (\text{by def. of } D_{\vec{u}})$$

$$= \lim_{t\to 0} \frac{1}{t}\Big(\int_{C+[Q,Q_t]} \vec{F}\cdot dr - \int_C \vec{F}\cdot dr \Big) \qquad (\text{by def of } \varphi)$$

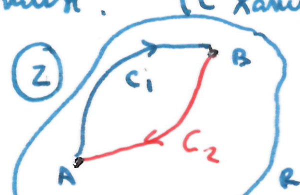
$$\underset{\text{additivity of } \int}{=} \lim_{t\to 0} \frac{1}{t} \int_{[Q,Q_t]} \vec{F}\cdot dr \underset{\substack{\text{unit tangent}\\ \text{to segment } = \vec{u}}}{=} \lim_{t\to 0} \frac{1}{t} \int_0^t F_{(x+su_1,y+su_2)}\cdot\vec{u}\, ds$$

$$= \lim_{t\to 0} \frac{1}{t}\Big(\int_0^t F_{(x+su_1,y+su_2)}\cdot\vec{u}\, ds - \underbrace{\int_0^0 F_{(x+su_1,y+su_2)}\cdot\vec{u}\, ds}_{=0 \ (\text{take } t=0)} \Big)$$

$$\underset{\text{def}}{=} \frac{d}{dt}\Big(\int_0^t F_{(x+su_1,y+su_2)}\cdot\vec{u}\, ds \Big)\Big|_{t=0} \underset{\text{Fund. Thm Calc.}}{=} \boxed{F_{(x,y)}\cdot\vec{u}}$$

Application: ① Use Theorem to show something is NOT conservative. Meaning, if we find 2 paths along which the integral has 2 values, then we know $\vec{F}$ is not conservative. (Examples in HW8)

②



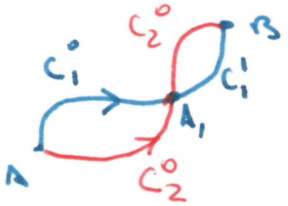
Theorem 3: Fix C piecewise smooth simple closed curve obtained by concatenating the curves C_1 & C_2 joining A & B (with opposite directions). Then, F is conservative if and only if $\oint_C \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot dr + \int_{C_2} \vec{F}\cdot dr$ equals 0 for every such C.

Proof: $\Rightarrow$) Use Theorem 1 & write $-C_2 = C_2^{op}$ = the curve C_2 w/ reverse orientation

By path independence: $$\int_{C_1} \vec{F}\cdot dr = \int_{C_2^{op}} \vec{F}\cdot dr \underset{\substack{\downarrow \\ \text{Lecture XXIX } (\vec{T}_{C_2^{op}} = -\vec{T}_{C_2})}}{=} -\int_{C_2} \vec{F}\, dr \quad (*)$$

So $$\oint_C \vec{F}\cdot dr = \int_{C_1} \vec{F}\, dr + \int_{C_2} \vec{F}\, dr \underset{\text{by } (*)}{=} 0.$$

$\Leftarrow$). Enough to prove independence of paths. Pick 2 piecewise smooth paths C_1 & C_2 joining A & B. If they intersect, break C_1 & C_2 along these N intersection pts



then we get $N+1$ points in R

$A = P_0$, $P_1, \ldots, P_{N-1}$, $P_N = B$

And curves $\begin{cases} C_1^i & \text{; joining } P_i \text{ with } P_{i+1} \\ C_2^i & \text{" } P_{i+1} \text{ " } P_i \end{cases}$

where C_1^i & C_2^i only meet at their ends.

We consider the simple closed curves C^i obtained by concatenating C_1^i & $(C_2^i)^{op}$

Then $$\oint_C \vec{F}\cdot dr = \sum_{i=0}^{N-1} \oint_{C^i} \vec{F}\cdot dr$$

$$\underset{\text{by hypothesis}}{0 =} \oint_{C^i} \vec{F}\cdot dr = \int_{C_1^i} \vec{F}\cdot dr + \int_{(C_2^i)^{op}} \vec{F}\, dr = \int_{C_1^i} \vec{F}\, dr - \int_{C_2^i} \vec{F}\, dr. \quad (**)$$

so $$\int_{C_1} \vec{F}\cdot dr \underset{\text{additivity}}{=} \sum_{i=0}^{N-1} \int_{C_1^i} \vec{F}\, dr \underset{\text{by } (**)}{=} \sum_{i=0}^{N-1} \int_{C_2^i} \vec{F}\, dr \underset{\text{additivity}}{=} \int_{C_2} \vec{F}\, dr$$

We conclude: the line integrals are path independent.

Since R is connected, by Thm 2 we conclude that $\vec{F}$ is conservative