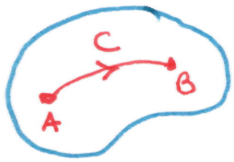
Lecture XXXI : §15.4 Green's Theorem

§0 Motivation (similar results in lower dimensions)

- Fund Thm of Calculus : f differentiable function $f:[a,b]\longrightarrow\mathbb{R}$. Then $\displaystyle\int_a^b \frac{df}{dx}\,dx = f(b)-f(a)$

  "Boundary of $[a,b]$" $=\{a,b\}$.

- Fund Thm for line integrals : $\varphi : D\longrightarrow\mathbb{R}$ diff'ble on a region D & a curve C in D from a pt A to a pt B, then $\displaystyle\int_C \nabla\varphi\cdot dr := \int_C \nabla\varphi\cdot\vec{T}\,ds = \varphi(B)-\varphi(A)$



"Boundary of C" $=\{A,B\}$

GOAL : Prove an analogous result for double integrals of conservative v. fields to line integrals along the curve bounding the region in the plane $\mathbb{R}^2$. (GREEN'S THM)

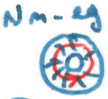
Write $\boxed{C : \vec{r}:[a,b]\longrightarrow\mathbb{R}^2 \quad \vec{r}(t)=\langle x(t),y(t)\rangle}$

Recall : $\begin{cases} \cdot\ C \text{ simple if it doesn't have self-closings } (\vec{r}(t_1)\neq\vec{r}(t_2) \text{ for all } a<t_1\neq t_2<b) \\ \cdot\ C \text{ closed if } \vec{r}(a)=\vec{r}(b)\end{cases}$

$\boxed{\text{TODAY : All our curves } C \text{ will be simple, closed \& piecewise smooth on a region } R \text{: connected \& simply connected}}$

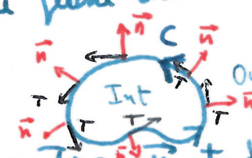
Nice regions : simply connected regions (: any curve in them can be contracted to a pt)

Non-eg :



eg

- Jordan's Thm (topological) : these curves divide the plane into 2 regions,
  - the interior : to the "left" of the curve
  - the outside : " " right of the curve



Int is simply connected! & bounded if C is oriented counterclockwise.

The outside contains the outward unit normal vectors $\vec{n}$ to the curve

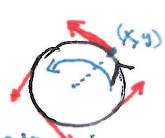
In coordinates : $\vec{T} = \dfrac{\langle x',y'\rangle}{\sqrt{x'^2+y'^2}}$, $\vec{n}=\dfrac{1}{\sqrt{x'^2+y'^2}}\langle y',-x'\rangle$

§1 Circulation form of Green's Thm : $F=\langle f,g\rangle$ v. field on R region in $\mathbb{R}^2$ containing C

Circulation $=\displaystyle\oint_C F\cdot dr = \oint_C F\cdot T\,ds = \int_a^b f\,x' + g\,y'\,dt \underset{\text{NOTATION}}{=} \oint_C f\,dx+g\,dy$.

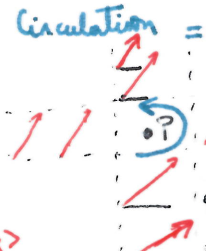
↳ measures the NET component of F in the tangent direction to C.

Eg : $F_{(x,y)}=\langle -y,x\rangle$ C = unit circle in $\mathbb{R}^2$ counterclockwise oriented



Circulation $=2\pi>0$ ($F=T$ in this case)

Why? Something inside the region is making it rotate counterclockwise



Around P.
$\uparrow$ f decreases : $\dfrac{\partial f}{\partial y}<0$
$\rightarrow$ g increases : $\dfrac{\partial g}{\partial x}>0$
$\Rightarrow$ curl <0

$F=\langle f,g\rangle$

curl <0 $\Rightarrow$ counterclockwise rotation at P.

Definition : The (2-dimensional) curl of $F=\langle f,g\rangle$ is

$$\boxed{\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}}$$

If curl is $=0$ on the region R, we say F is irrotational on R

Theorem (Green's Thm in Circulation Form)

Fix C simple, closed, piecewise-smooth curve counterclockwise oriented and enclosing a region R in $\mathbb{R}^2$ that is connected & simply connected. Write $F = \langle f, g \rangle$ v.field on R & assume f & g have continuous 1st partial derivatives in R. Then:

$$\oint_C F\cdot dr = \oint_C f\,dx + g\,dy = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{\text{curl}} dA$$

Back to example: curl of F: $1-(-1) = 2 > 0$ $\quad \oint F\,dr = \iint_R 2\,dA = 2\cdot \text{Area}(R) = \boxed{2\pi}$

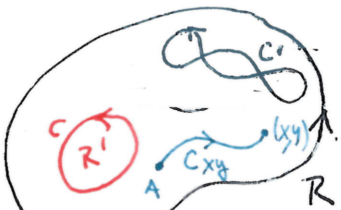
Applications:

1) Last time: R is conn & simply conn; F w/ cont first partials. F conservative ($\nabla\varphi = F$ for some φ) $\Leftrightarrow$ $f_y = g_x$ on R $\Leftrightarrow$ curl $= 0$ on R.

φ_{xy} $\quad \varphi_{yx}$

Proof: ($\Rightarrow$) required nothing about R

($\Leftarrow$) Follows from Green's Theorem because $\oint_C F\cdot dr = \iint_{R'} \underset{\text{curl}}{0}\,dA = 0$ for any simple closed curve



so the same is true for any closed curve C' in R (break it into simple pieces) and we know this ensures path independence of F & hence we can build a potential function for F:

$$\varphi(x,y) := \int F\cdot dr \quad \text{for any curve } C_{xy} \text{ joining } A \text{ with a pt } (x,y)$$

Application ②: Calculation of areas enclosed by simple closed curves

Pick 2 special v.fields
$$\begin{cases} \vec{F_1} = \langle 0, x\rangle \rightsquigarrow \text{curl } F_1 = 1 - 0 = 1 \\ \vec{F_2} = \langle y, 0\rangle \rightsquigarrow \text{curl } F_2 = 0 - 1 = -1. \end{cases}$$

By Green's Thm: $\text{Area}(R) = \iint_R 1\,dA = \oint_C \langle 0,x\rangle\cdot dr = \oint_C x\,dy$

$$-\text{Area}(R) = \iint_R -1\,dA = \oint_C \langle y,0\rangle\cdot dr = \oint_C y\,dx$$

Thm: $\text{Area}(R) = \oint_C x\,dy = \oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$

Example: Area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad (a,b>0)$

- Earlier: $x = ar\cos\theta$, $y = br\sin\theta$ & use change of variables $J(r,\theta) = abr$ $\quad 0\le r\le 1$, $0\le\theta\le 2\pi$ $\quad A(R) = \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta$

- Using Green's Thm: On boundary: $r=1$, so $F = \langle a\cos\theta, b\sin\theta\rangle$

$$x\,dy - y\,dx = a\cos\theta\, b\cos\theta\,d\theta - b\sin\theta(-a\sin\theta)\,d\theta = ab\,d\theta.$$

$$\text{so Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \boxed{ab\pi}$$

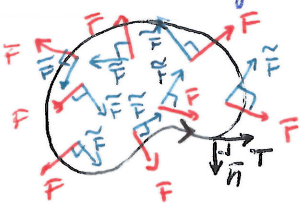
§ 2 Flux form of Green's Thm: $F = \langle f, g\rangle$ $\quad \vec{n} = \dfrac{\langle y', -x'\rangle}{|\vec{r}\,'(t)|}$

Flux: $\displaystyle\oint_C F\cdot\vec{n}\,ds = \int_a^b F(x(t),y(t))\cdot\langle y', -x'\rangle\,dt = \oint_C f\,dy - g\,dx$

This is the circulation of $\tilde{F} = \langle -g, f\rangle$ so we get a new version of Green's Theorem.

$$\operatorname{curl}\tilde{F} = f_x - (-g)_y = \boxed{f_x + g_y =: \text{divergence of } F}$$

• What does the divergence measure?



$\tilde{F} \perp F$

• If $\operatorname{div}(F) > 0 \Rightarrow \tilde{F}$ has >0 circulation along C
so F has >0 outward flux
Think of the Interior of C as a source for the flow of F

• Similarly: $\operatorname{div}(F) < 0 \Rightarrow \operatorname{Int}(C)$ is a sink for the flow of F.

Def: If divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$ on R, we say the vector field is source free on R.

Green's Thm (Flux Form): R simply conn region bounded by simple closed curve C

$$\oint_C F\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA.$$

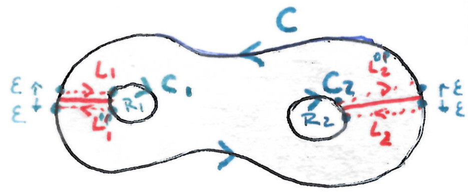
[Thm has 2 folded application: compute line integrals via (easier) divergence double integrals & conversely, compute double integrals via (easier) line integrals. We need to find suitable f & g giving the known expression $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$. (examples in HW9 & Recitation 11)

§ 3 More general regions? R connected region in $\mathbb{R}^2$

If R Not simply connected; cut it w/ curves to make it into a simply connected region bounded by a closed curve.

$\boxed{C^{op} = C \text{ w/ reverse orientation}}$

Example



R has 2 holes. → make 2 cuts to join 2 inner curves to out most curve.
boundary of R = 3 curves.
Break C into 2 pieces (as many as the # cuts)

New curve: $C^{top} + L_1 + C_1 + L_1^{op} + C^{bot} + L_2 + C_2 + L_2^{op}$, New region $R'_\varepsilon = R \setminus \{L_1^{(\varepsilon)} \cup L_2^{(\varepsilon)}\}$ is simply conn.

Integrals add up & $\displaystyle\oint_{L_i} F\cdot\eta\,ds = -\oint_{L_i^{op}} F\cdot\eta\,ds$ for $i=1,2$. ⇝ cancel out

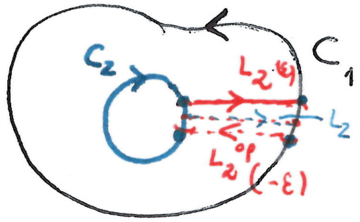
Take limit as $\varepsilon \to 0$:

$$\oint_C F\cdot\vec{\eta}\,ds + \oint_{C_1} F\cdot\vec{\eta}\,ds + \oint_{C_2} F\cdot\vec{\eta} = \oint_C F\cdot\vec{\eta}\,ds - \oint_{C_1^{op}} F\cdot\vec{\eta}\,ds - \oint_{C_2^{op}} F\cdot\vec{\eta}\,ds$$

$$\xrightarrow{\text{Green's Thm.}} = \iint_{R} \operatorname{div}(F)\,dA - \iint_{R_1} \operatorname{div}(F)\,dA - \iint_{R_2} \operatorname{div}(F)\,dA.$$

→ Notice: Orient the curves so that the region is to the left of the curves.

Easier example



$C = C_1 + C_2$

We remove the strip of width 2ε from R.
$\Rightarrow$ the new region $R'_{(\varepsilon)}$ is connected, simply connected & bounded by a piecewise smooth curve C' that is closed & simple

$$C' = C_1(\varepsilon) + L_2^{op}(-\varepsilon) + C_{2(\varepsilon)} + L_2(\varepsilon)$$

Note

$$\int_{C_1(\varepsilon)} F\cdot dr \longrightarrow \oint_{C_1} F\cdot dr \quad ; \quad \int_{L_2^{op}(-\varepsilon)} F\cdot dr \longrightarrow \int_{L_2^{op}} F\cdot dr$$

$$\int_{C_2(\varepsilon)} F\cdot dr \longrightarrow \oint_{C_2} F\cdot dr = -\oint_{C_2^{op}} F\cdot dr \quad ; \quad \int_{L_2(\varepsilon)} F\cdot dr \longrightarrow \int_{L_2} F\cdot dr = -\int_{L_2^{op}} F dr$$

By Green's Thm on C' & the region R':

$$\int_{C'} F\cdot dr = \iint_{R'(\varepsilon)} \operatorname{curl}(F)\, dA \xrightarrow[\varepsilon\to 0]{} \iint_R \operatorname{curl}(F)\, dA = \iint \operatorname{curl}(F)\, dA - \iint \operatorname{curl}(F)\, dA$$

$$\parallel$$

$$\int_{C_1(\varepsilon)} F\cdot dr + \int_{C_2(\varepsilon)} F\cdot dr + \int_{L_2^{op}(-\varepsilon)} F\cdot dr + \int_{L_2^{op}(-\varepsilon)} F\cdot dr \xrightarrow[\varepsilon\to 0]{} \oint_{C_1} F\cdot dr - \oint_{C_2^{op}} F\cdot dr + 0$$

Conclusion: $\displaystyle\oint_C F\cdot dr = \oint_{C_1} F\cdot dr - \oint_{C_2^{op}} F\cdot dr = \iint_R \operatorname{curl}(F)\, dr$

It's very important to have the correct orientations for the 2 curves so that the region is always at the left of the curves (with respect to their orientation).

§4.5 Stream functions:

INPUT : $\vec{F} = \langle f, g\rangle : R \longrightarrow \mathbb{R}^2$ vector field & f, g with cont. 1^{st} partials

Def: A function $\Psi : R \longrightarrow \mathbb{R}$ is a stream function for $\vec{F}$ if $\frac{\partial\Psi}{\partial y} = f$ & $\frac{\partial\Psi}{\partial x} = g$

Why? ① $\operatorname{div}(F) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} \underset{\Psi \text{ stream function}}{=} \dfrac{\partial}{\partial x}\left(\dfrac{\partial\Psi}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(-\dfrac{\partial\Psi}{\partial x}\right) = \Psi_{yx} - \Psi_{xy} = 0$

② Flow curves are level curves of stream function $\Psi_{(x,y)} = $ constant.
By implicit differentiation on a parametrization $\langle x(t), y(t)\rangle$ of the flow curve

$$0 = \frac{d}{dt}\left(\Psi(x(t), y(t))\right) = \Psi_x\, x' + \Psi_y\, y' = \underbrace{\langle y', -x'\rangle}_{\text{outer normal}} \cdot \underbrace{\left\langle \frac{\partial\Psi}{\partial y}, \frac{-\partial\Psi}{\partial x}\right\rangle}_{=\vec{F}}$$

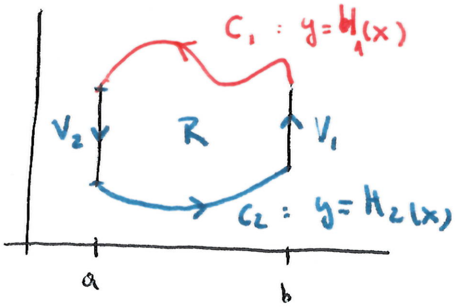
so $\vec{F}$ is tangent to the flow curve.

Thm: If R conn & simply conn. Then: $\vec{F}$ admits a stream function $\Longleftrightarrow$ $\operatorname{div}(F) = 0$

§5 Proof of Green's Theorem (circulation form)

Recall, we assume $\vec{F} = \langle f, g \rangle$ has continuous first partial derivatives. We prove the result in 3 steps:

① Proof for Type I regions: We show $\oint_C f\,dx = \iint_R -\frac{\partial f}{\partial y}\,dA$.



$$\iint_R \frac{\partial f}{\partial y}\,dA \underset{\text{Fubini}}{=} \int_a^b \left(\int_{H_2(x)}^{H_1(x)} \frac{\partial f}{\partial y}_{(x,y)}\,dy \right) dx = \int_a^b f(x,y)\Big|_{y=H_2(x)}^{y=H_1(x)}\,dx$$

$$= \int_a^b f(x, H_1(x)) - f(x, H_2(x))\,dx$$

$$\oint_C f\,dx = \int_{C_1} f\,dx + \int_{V_2} f\,dx + \int_{C_2} f\,dx + \int_{V_1} f\,dx$$

$C_1^{op} : \vec{r}_1 : [a,b] \to \mathbb{R}^2$ $\quad \vec{r}_1(t) = \langle t, H_1(t) \rangle$, $\vec{r}_1{}'(t) = \langle 1, H_1'(t) \rangle$ $\rightsquigarrow \int_{C_1} f\,dx = -\int_{C_1^{op}} f\,dx = \int_a^b f(t, H_1(t)) \cdot 1\,dt$

$V_2^{op} : \vec{r}_{v_2} : [H_2(a), H_1(a)] \to \mathbb{R}^2$ $\quad \vec{r}_{v_2}(t) = \langle a, t \rangle$ $\rightsquigarrow \int_{V_2} f\,dx = -\int_{V_2^{op}} f\,dx = 0$
$\vec{r}\,'_{v_2}(t) = \langle 0, 1 \rangle$

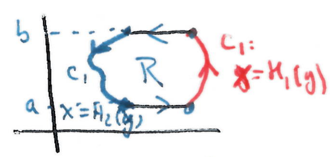
$V_2 : \vec{r}_2\ [a,b] \to \mathbb{R}^2$ $\quad \vec{r}_2(t) = \langle t, H_2(t) \rangle$ $\rightsquigarrow \int_{C_2} f\,dx = \int_{a_2}^{b} f(t, H_2(t)) \cdot 1\,dt$
$\vec{r}_2{}'(t) = \langle 1, H_2'(t) \rangle$

$V_1 : \vec{r}_{v_1} : [H_2(b), H_1(b)] \to \mathbb{R}^2$ $\quad \vec{r}_{v_1}(t) = \langle b, t \rangle$ $\rightsquigarrow \int_{V_1} f\,dx = 0$

So $\boxed{\oint_C f\,dx} = \int_a^b f(t, H_2(t))\,dt - \int_a^b f(t, H_1(t))\,dt = \int_a^b f(t,H_2(t)) - f(t, H_1(t))\,dt$

$$= \boxed{\iint_R -\frac{\partial f}{\partial y}\,dA}$$

② Proof for Type II regions: We show by symmetry with ① that $\oint_C g\,dy = \iint_R \frac{\partial g}{\partial x}\,dA$ for a Type II region R

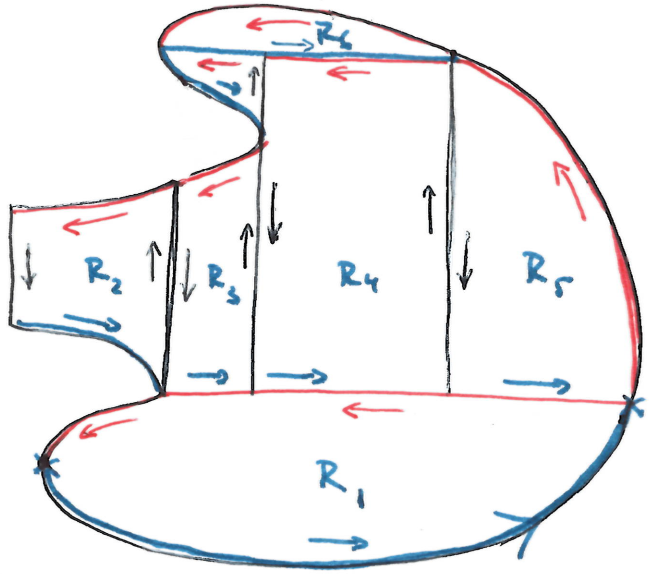


③ Proof for general regions: We decompose R into a finite collection of Type I regions ONLY to show $\oint_C F \cdot dx = \iint_R -\frac{\partial f}{\partial y}(x,y)\,dA$. Similarly we decompose into type II regions only to show $\oint_C g \cdot dy = \iint_R \frac{\partial g}{\partial x}(x,y)\,dA$

We add these 2 identities to get $\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

Decomposition idea:

Decomp into Type I reg.



On each R_i we have: $\iint_{R_i} -\frac{\partial f}{\partial y}\,dA = \oint_{\text{boundary of } R_i} f\,dx$

Note: • All the internal horiz/vertical segments are traversed twice, once in each direction, so these line integrals cancel out.

• The boundary curve C is broken into pieces & traversed counter-clockwise, as required.

• Adding all double integrals over the 6 regions gives $\iint_R -\frac{\partial f}{\partial y}\,dA$

• Adding all 6 line integrals along the boundaries of the 6 regions $R_1, \dots, R_6$ gives $\oint_C f\,dx$.

• Conclusion: $\iint_R -\frac{\partial f}{\partial y}\,dA = \oint_C f\,dx$ for all open & simply connected regions R bounded by a closed, simple & piecewise smooth curve C.