Lecture XXXII: §15.5 Divergence & curl

GOAL for the remaining of the course: lift two forms of Green's Thm from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Recall:
- R conn, simply connected region in $\mathbb{R}^2$ bounded by a piecewise smooth, simple, closed curve C oriented counterclockwise
- $F = \langle f, g\rangle$ v. field on $\mathbb{R}^2$; f, g with cont. 1st partials

① Green's Thm in circulation form:

$$\mathrm{Circ}(F) = \oint_C F\cdot T\, ds = \iint_R \underbrace{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}_{=\,\mathrm{curl}(F)}\, dA$$

T = unit tangent

② Green's Theorem in flux form:

$$\mathrm{Flux}(F) = \oint_C F\cdot n\, ds = \iint_R \underbrace{\frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}}_{=\,\text{divergence}(F)}\, dA$$

$\vec{n}$ = unit outer normal

How we lift these results:

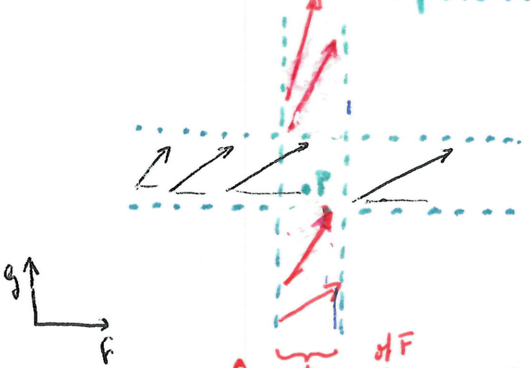
Ⓐ replace line integrals by surface integrals

Ⓑ extend curl & divergence operators to vector fields in $\mathbb{R}^3$. ← TODAY's TOPIC

§1 The Divergence:

Definition: The divergence of $\vec{F} = \langle f, g, h\rangle$ v. field in $\mathbb{R}^3$ is $\boxed{\mathrm{div}(F)(x,y,z) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}}$ → [$\mathbb{R}$-scalar function]

It measures the expansion or contraction of the vector field at each point.



} f = 1st component of F increases for fixed y } $\Longleftrightarrow f_x > 0$

g = 2nd comp. of F increases for fixed x } $\Longleftrightarrow g_y > 0$

so $\mathrm{div} = f_x + g_y > 0$ at P, indicating outward expansion. The converse also holds.

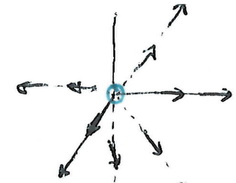
We use the following notation ∇ = del operator $= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$

↳ we evaluate it on functions.

So $\nabla\cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle f, g, h\rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \mathrm{div}(F)$.

Examples: ① $F = \langle x, y, z^2\rangle \rightsquigarrow \mathrm{div}(F) = \nabla\cdot F = 1 + 1 + 2z = 2 + 2z$.

② $F = \dfrac{r}{|r|} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$ radial v. field defined in $\mathbb{R}^3 - \{(0,0,0)\}$



$$\frac{\partial f}{\partial x} \underset{\text{Quot Rule}}{=} \frac{1\sqrt{x^2+y^2+z^2} - x\,(x^2+y^2+z^2)^{-1/2}\frac{2x}{2}}{(x^2+y^2+z^2)} = \frac{(x^2+y^2+z^2) - x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{|r|^2-x^2}{|r|^3}$$

By symmetry $\dfrac{\partial g}{\partial y} = \dfrac{|r|^2-y^2}{|r|^3}$ & $\dfrac{\partial h}{\partial z} = \dfrac{|r|^2-z^2}{|r|^3}$

So $\operatorname{div}(F) = \dfrac{3|r|^2-|r|^2}{|r|^3} = \boxed{\dfrac{2}{|r|}} > 0$ so F is expanding outwards.

Theorem: Divergence of Radial v. fields

If $\vec{F} = \dfrac{r}{|r|^p} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}}$, then $\operatorname{div}(F) = \dfrac{3-p}{|r|^p}$

Pf/. If $p=0$ $\vec{F} = \langle x, y, z\rangle$ $\operatorname{div} = 3 = \dfrac{3-0}{|r|^0}$. ✓

We differentiate $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{p/2}}\right) = \dfrac{|r|^p - x\,|r|^{p-2}\,\frac{p}{2}\cdot 2x}{|r|^{2p}}$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{|r|^{p-2}\,(|r|^2 - px^2)}{|r|^{2p}} = \frac{|r|^2-px^2}{|r|^{p+2}}$$

By symmetry: $\dfrac{\partial g}{\partial y} = \dfrac{|r|^2-py^2}{|r|^p}$, $\dfrac{\partial h}{\partial z} = \dfrac{|r|^2-pz^2}{|r|^p}$.

So $\operatorname{div}(F) = \dfrac{3|r|^2 - p|r|^2}{|r|^{p+2}} = \boxed{\dfrac{3-p}{|r|^p}}$

Note: In particular $\operatorname{div}(F) = 0 \iff p=3$.

§2 Curl:

To generalize the curl $= g_x - f_y$ from $\mathbb{R}^2$ to $\mathbb{R}^3$, we use $\nabla\times F$:

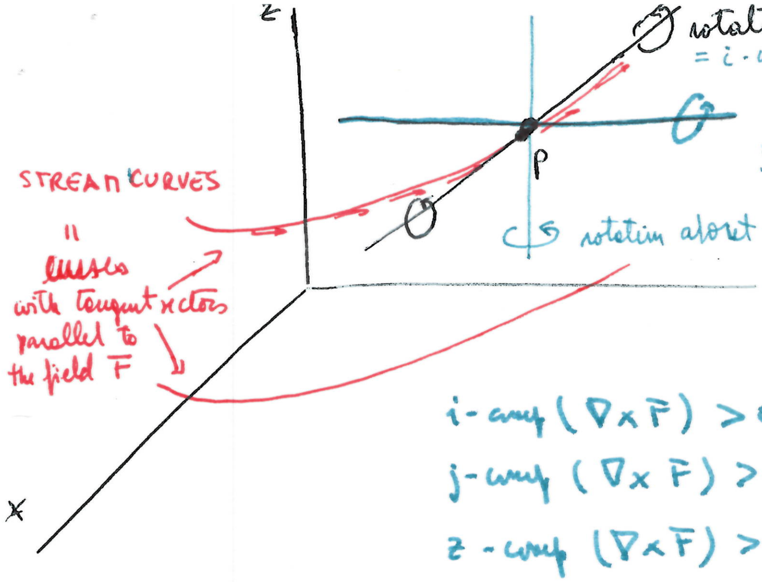
Def Write $F = \langle f, g, h\rangle$. The curl of F is defined by the formula

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) - j\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) + k\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

Recall: Curl in $\mathbb{R}^2 = g_x - f_y$ We view it as $k\left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right)$ where $F = \langle \underset{(x,y,z)}{f}, \underset{(x,y)}{g}, \underset{(x,y)}{\overset{h}{\overset{\shortparallel}{0}}}\rangle$.

k-comp of Curl gives the rotation of F in the xy-plane at a point. $\boxed{\text{curl} > 0 \text{ in } \mathbb{R}^2 = \text{COUNTERCLOCKWISE rotation}}$

Thus: i- & j-comp. of curl give the rotation of F in the yz- & xz-planes respectively.



rotation on yz-plane = rotation about axis parallel to x-axis
$= i$-comp $(\nabla \times F)$

rotation about axis parallel to y-axis
j-comp of $\nabla \times F$

rotation about axis parallel to z-axis $= k$-comp of $\nabla \times F$

i-comp $(\nabla \times F) > 0 \Rightarrow$ counterclockwise rotation wrt x-axis

j-comp $(\nabla \times F) > 0 \Rightarrow$ ———————— y-axis

z-comp $(\nabla \times F) > 0 \Rightarrow$ ———————— z-axis

Def If $\nabla \times F = \vec{0}$, we say the vector field is irrotational (in $\mathbb{R}^2$ the circulation was $= 0$).

§3. Curl of a general Rotation Vector Field:

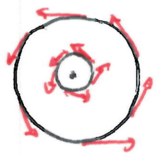
Special vector fields $\vec{F}_{(x,y,z)} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is a constant vector $\vec{a} \neq \vec{0}$

& $\vec{r} = \langle x, y, z \rangle$. In coordinates:

$$\vec{F} = \vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x \rangle$$

It's called a general rotational field in $\mathbb{R}^3$

Eg: $a_1 = a_2 = 0$, $a_3 = 1$, we get $\vec{F} = \langle -y, x, 0 \rangle = \langle -y, x \rangle$ usual rotational v.field in $\mathbb{R}^2$

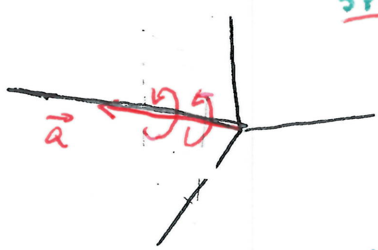


axis of rotation = z-axis
counterclockwise rotation if looking from the positive z-axis.

In general: $\vec{F}$ is a rotational field with axis of rotation in the direction of $\vec{a}$.

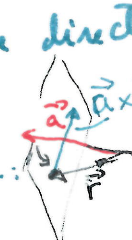
3 Properties:

① $\nabla \cdot \vec{F} = \underbrace{\frac{\partial}{\partial x}(a_2 z - a_3 y)}_{=0} + 0 + 0 = 0$. So $\boxed{\operatorname{div}(\vec{F}) = 0}$



② $\vec{F} = \vec{a} \times \vec{r}$ circles the vector $\vec{a}$ in the counterclockwise direction looking along $\vec{a}$ from head to tail.

Why? $\vec{F}$ lies in orthogonal plane to $\vec{a}$. & use right-hand rule:



③ $\nabla \times F = i\,(a_1 - (-a_1)) - j\,(-a_2 - a_2) + k\,(a_3 - (-a_3)) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\vec{a}$.

$\begin{cases} f = a_2 z - a_3 y \\ g = a_3 x - a_1 z \\ h = a_1 y - a_2 x \end{cases}$ & use definition of $\nabla \times F$.

- So curl$(\vec{F})$ has the same direction as the axis of rotation $(\vec{a})$.
- $|\text{curl}(\vec{F})| = |\nabla \times F| = 2|\vec{a}|$

- $|\vec{a}|$ is the constant angular speed of rotation = rate at which a particle rotates about the axis of the field

Q: Suppose a paddle wheel is placed in $\vec{F}$ at a pt P with the axis of the wheel in the direction of a unit vector $\vec{n}$. For which $\vec{n}$'s does the paddle spin faster?

$$\text{comp}_{\vec{n}}\, \nabla\times\vec{F} = \frac{(\nabla\times\vec{F})\cdot\vec{n}}{\underbrace{|\vec{n}|}_{=1}} = |\nabla\times\vec{F}|\cdot|\vec{n}|\cos\theta$$ has to be maximal.

So $\cos\theta = 1$, ie $\theta = 0$. Conclusion: Fastest speed when $\vec{n}$ is parallel to $\nabla\times\vec{F}$

If $\theta = \pm\frac{\pi}{2}$, then no rotation at all. (the wheel doesn't spin)

§4 Divergences & curl properties

Recall:

div $\begin{cases} \nabla\cdot(F+G) = \nabla\cdot F + \nabla G & \rightsquigarrow \text{div}(F+G) = \text{div}(F) + \text{div}(G) \\ \nabla\cdot(cF) = c\,\nabla\cdot F & \rightsquigarrow \text{div}(cF) = c\,\text{div}(F). \end{cases}$

curl $\begin{cases} \nabla\times(F+G) = \nabla\times F + \nabla\times G & \rightsquigarrow \text{curl}(F+G) = \text{curl}(F) + \text{curl}(G) \\ \nabla\times(cF) = c\nabla\times F & \rightsquigarrow \text{curl}(cF) = c\,\text{curl}(F). \end{cases}$

Thm 1: Suppose F is a conservative v. field on an open region D of $\mathbb{R}^3$ & whose potential function φ has cont 2^{nd} order partials on D. Then, $\nabla\times F = \nabla\times\nabla\varphi = \vec{0}$ (curl(F) $=\vec{0}$)

Pf/ $$\nabla\times\nabla\varphi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = \underbrace{(\varphi_{zy}-\varphi_{yz})}_{=0 \text{ mixed partials are cont.}} i - j\underbrace{(\varphi_{zx}-\varphi_{xz})}_{=0} + k\underbrace{(\varphi_{yx}-\varphi_{xy})}_{0} = \vec{0}$$

Note: If D is connected & simply connected, the converse is also true (see §15.7 Stoke's Thm)

Thm 2: If $F = \langle f, g, h\rangle$ where f, g, h have cont. 2^{nd} partials, then $\nabla\cdot\underbrace{(\nabla\times F)}_{\text{v. field}} = 0$ meaning div (curl(F)) $= 0$.

Pf/ $$\nabla\cdot(\nabla\times F) = \frac{\partial}{\partial x}(h_y - g_z) + \frac{\partial}{\partial y}(f_z - h_x) + \frac{\partial}{\partial z}(g_x - f_y) = (h_{yx} - h_{xy}) + (f_{zy} - f_{yz}) + (g_{xz} - g_{zx}) = 0$$

because all mixed partials are cont, so they agree.

Thm 3 (Prod Rule for divergence)

α scalar function differentiable on D & $F: D \to \mathbb{R}^3$ differentiable v. field. Then

$$\nabla\cdot(\alpha F) = \nabla\alpha\cdot F + \alpha(\nabla\cdot F).$$ is a scalar function.