Lecture XXXIII: §15.6 Surface integrals

Motivating example: Calculate the average temperature on the surface of a sphere: need to "add up" the continuous Temp values on the sphere & divide by the surface area.

⇓

surface integrals!

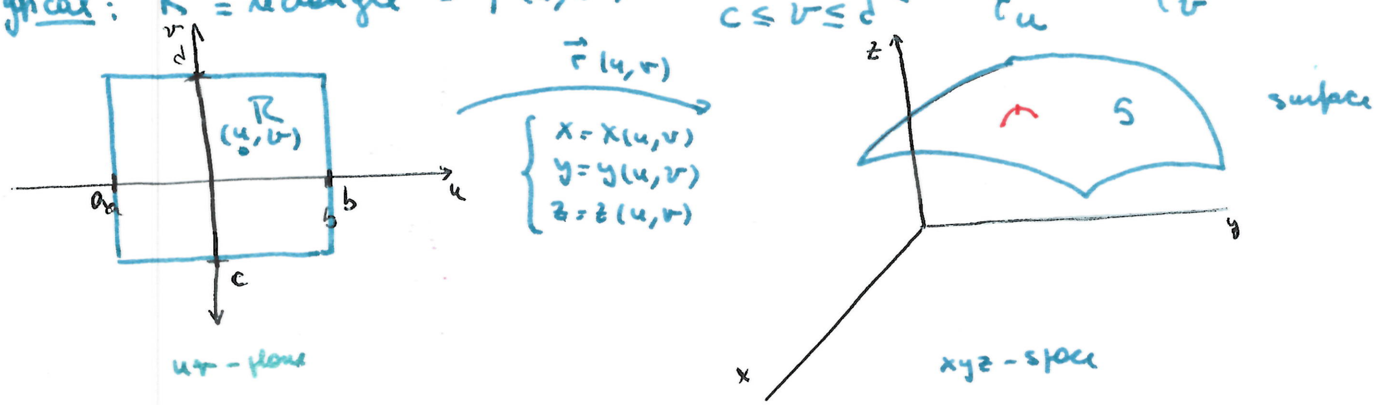
§1 Parameterized surfaces

Recall: Parameterized curves C : $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle \quad a \le t \le b$. (1 parameter)

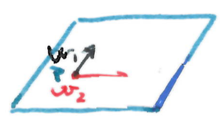
A Surface is an object of dimension 2, so we need 2 parameters to define it:

$$\boxed{\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle \quad \text{where } (u,v) \text{ in } R \text{ region in } \mathbb{R}^2}$$

Typical: R = rectangle $= \left\{(u,v) : \begin{matrix} a \le u \le b \\ c \le v \le d\end{matrix}\right\} = \underset{u}{[a,b]} \times \underset{v}{[c,d]}$.



Examples: ⓐ Planes



$\vec{w}_1 = \langle a_1, b_1, c_1\rangle$

$\vec{w}_2 = \langle a_2, b_2, c_2\rangle$

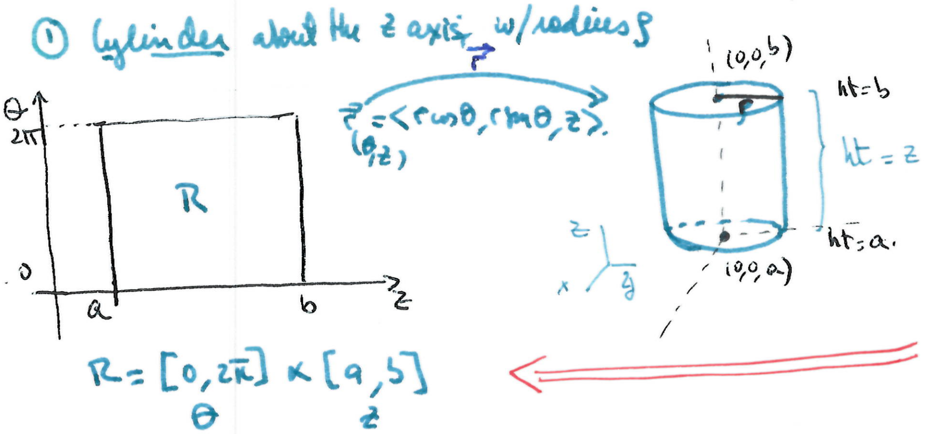
$P = (p_1, p_2, p_3)$.

Plane has vector eqn: $(x,y,z) = P + t\,\vec{w}_1 + s\,\vec{w}_2$

$$\vec{r}(s,t) = \langle \underbrace{p_1 + t a_1 + s a_2}_{= x(s,t)},\ \underbrace{p_2 + t b_1 + s b_2}_{= y(s,t)},\ \underbrace{p_3 + t c_1 + s c_2}_{= z(s,t)}\rangle$$

$R = \mathbb{R}^2$

① Cylinder about the z axis w/ radius ρ



$$\left\{\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z \end{aligned}\right. \qquad \begin{aligned} &0 \le \theta \le 2\pi \\ &\rho \text{ fixed} \\ &a \le z \le b \end{aligned}$$

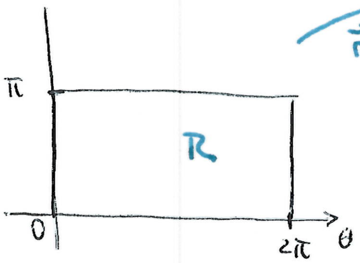
⇓

$\boxed{\text{parameters} = (\theta, z)}$

(angle & ht)

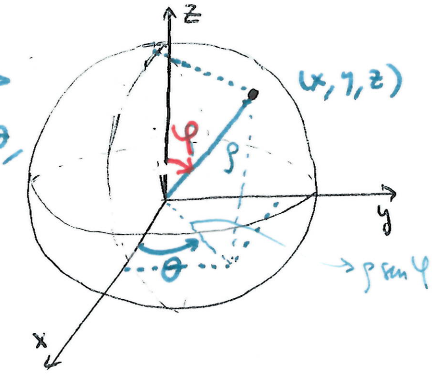
$R = \underset{\theta}{[0, 2\pi]} \times \underset{z}{[a,b]}$

② Spheres: centered at the origin w/ radius ρ (fixed) $\Rightarrow$ spherical coords



$\vec{r}(\theta,\varphi)$

$\vec{r}(\theta,\varphi) = \langle \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi \rangle$



$$\begin{cases} x = (\rho\sin\varphi)\cos\theta \\ y = \rho\sin\varphi\ \sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

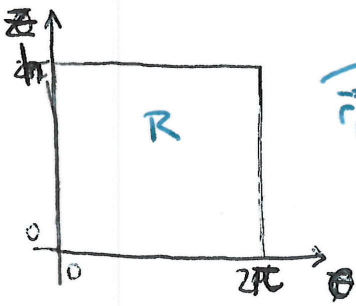
ρ fixed, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

$\Downarrow$

$R = \underset{\theta}{[0, 2\pi]} \times \underset{\varphi}{[0, \pi]}$ $\Longleftarrow$ $\boxed{\text{parameters} = (\theta, \varphi)}$

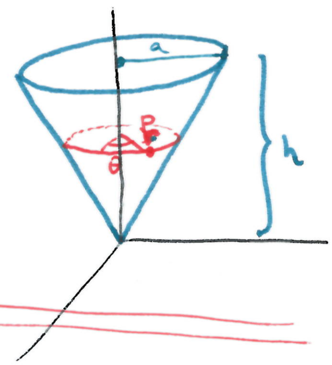
(2 angles)

③ Cones of height h & radius a with vertex at $(0,0,0)$: Equation: $z = \frac{h}{a}(x^2+y^2)$



$\vec{r}$

$\vec{r}(\theta, z) = \langle \frac{az}{h}\cos\theta, \frac{az}{h}\sin\theta, z \rangle$



$$\begin{cases} x = \rho\cos\theta & 0 \le \rho \le a \\ y = \rho\sin\theta & 0 \le \theta \le 2\pi \\ z = \frac{h}{a}\rho & h, a \text{ fixed} \end{cases}$$

$\Downarrow$

$\Longleftarrow$ $\boxed{\text{parameters} = (\theta, z)}$

(angle & ht)

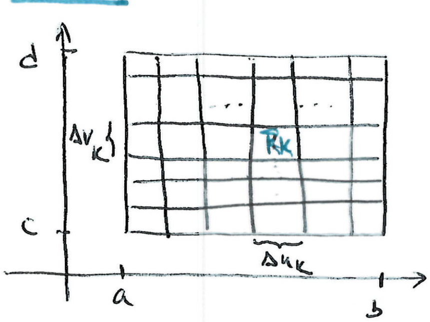
§2 Surface integrals of scalar-valued functions

$\begin{cases} f = f_{(x,y,z)} : S \longrightarrow \mathbb{R} \quad \text{function defined on the surface } S. \\ \text{Parameterization of } S: \vec{r}_{(u,v)} = \langle x(u,v), y(u,v), z(u,v) \rangle \quad \text{where } (u,v) \text{ on } R \ \& \end{cases}$

R is a rectangle $= \underset{u}{[a,b]} \times \underset{v}{[c,d]}$

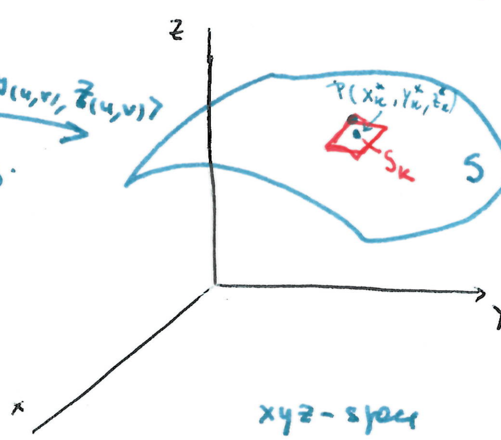
GOAL: Define $\iint_S f_{(x,y,z)}\, dS$ ↳ differential wrt. surface area.

Method: Use Riemann sums.



$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Parameterize.



$(x_k^*, y_k^*, z_k^*) = \vec{r}(u_k^*, v_k^*)$

uv-parameter space   xyz-space

R_k rectangle $\longrightarrow$ S_k surface

STEP 1: Partition R into N rectangles ~~with~~ sides parallel to the u- & v-axes & the rectangle R_k has lengths Δu_k & Δv_k, respectively

$\Rightarrow \text{Area}(R_k) = \Delta u_k \Delta v_k$.

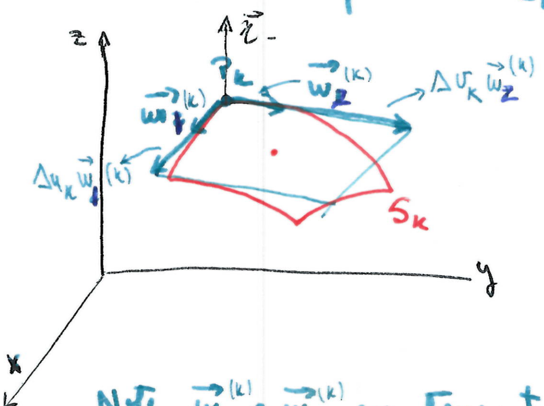
R_k maps to a surface patch S_k of area $\Delta S_k = ?$.

STEP 2: Pick a point $Q_k = (u_k^*, v_k^*)$ in R_k $\xrightarrow{\ r\ }$ $(x_k^*, y_k^*, z_k^*) = \vec{r}(u_k^*, v_k^*)$ is a point in S_k.

We define the Riemann sum:

$$\sum_{i=1}^{N} f\left(x(u_k^*, v_k^*), y(u_k^*, v_k^*), z(u_k^*, v_k^*)\right) \cdot \Delta S_k \qquad (*)$$

STEP 3: Compute ΔS_k area of the patch.



Idea: approximate S_k by the tangent plane at the marked point $(x_k^*, y_k^*, z_k^*) = P_k$ (Assume this plane exists at P_k)

We draw the picture when (x_k^*, y_k^*, z_k^*) is the top-left "corner" of the surface patch.

Area of tangent parallelogram $= |\vec{w}_1 \times \vec{w}_2|$

Note: $\vec{w}_1^{(k)}$ & $\vec{w}_2^{(k)}$ are tangent vectors to S_k at the pt P_k.

$\vec{w}_1^{(k)}$ = tangent vector to S corresponding to a change in u with v constant in the uv-plane

$\vec{w}_2^{(k)}$ = ——————————— v —— u ———————

$$\Rightarrow \vec{w}_2^{(k)} = \frac{\partial \vec{r}}{\partial v}(Q_k) = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle (Q_k) \qquad \vec{w}_1^{(k)} = \frac{\partial \vec{r}}{\partial u}(Q_k) = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle (Q_k).$$

By linear approximation: $\frac{\partial \vec{r}}{\partial u}(Q_k) \approx \frac{1}{\Delta u}\left(\vec{r}(u_k^* + \Delta u_k, v_k^*) - \vec{r}(u_k^*, v_k^*)\right)$

so $\Delta u_k \vec{w}_1^{(k)} \approx \vec{r}(u_k^* + \Delta u_k, v_k^*) - \vec{r}(u_k^*, v_k^*)$

Similarly, $\Delta v_k \vec{w}_2^{(k)} \approx \vec{r}(u_k^*, v_k^* + \Delta v_k) - \vec{r}(u_k^*, v_k^*)$

$\Rightarrow$ Parallelogram approximating S_k has sides $\Delta u_k \vec{w}_1^{(k)}$ & $\Delta v_k \vec{w}_2^{(k)}$

$$\text{Area}(S_k) \approx |\Delta u_k \vec{w}_1^{(k)} \times \Delta v_k \vec{w}_2^{(k)}| = \Delta u_k \Delta v_k \, \underbrace{|\vec{w}_1^{(k)} \times \vec{w}_2^{(k)}|}$$

must be $\neq \vec{0}$ (otherwise the Area(S_k) is badly approxim.

Replace this in $(*)$ and take limits as $\Delta u = \max_k \Delta u_k \longrightarrow 0$

$\Delta v = \max_k \Delta v_k \longrightarrow 0$.

similar story if we pick u_k^*, v_k^* to be any point in R_k.

Definition $$\iint_S f_{(x,y,z)}\,dS = \lim_{(\Delta u,\Delta v)\to(0,0)} \sum_{k=1}^{n} f\big(x(u_k^*,v_k^*),\, y(u_k^*,v_k^*),\, z(u_k^*,v_k^*)\big) \left|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right|_{(u_k^*,v_k^*)} \Delta u\,\Delta v$$

$$= \iint_R f\big(x(u,v),\, y(u,v),\, z(u,v)\big) \left|\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}\right| dA$$

where $\frac{\partial \vec r}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial v}\right\rangle$ & $\frac{\partial \vec r}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$. are assumed to be continuous & $\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v} \neq 0$ on R.

Note: The same definition will work for R = domain of $\vec r_{(u,v)}$ not necessarily a rectangle

Examples: ① Cylinder $\vec r_{(\theta,z)}\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z\end{cases}$ $\Rightarrow$ $\frac{\partial \vec r}{\partial \theta} = \langle -\rho\sin\theta, \rho\cos\theta, 0\rangle$

$\frac{\partial \vec r}{\partial z} = \langle 0, 0, 1\rangle$

$f = 1$: $\quad \text{Area}(S) = \iint_S 1\, dS = \int_a^b\int_0^{2\pi} 1\ \rho\, d\theta\, dh = 2\pi\rho\underbrace{(b-a)}_{\text{ht.}}$

$\begin{vmatrix} i & j & k \\ -\rho\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1\end{vmatrix} = \langle \rho\cos\theta, \rho\sin\theta, 0\rangle \Rightarrow \text{magnitude} = |\rho| = \rho > 0$

§3 Surface integrals for graphs

Notice: graph of a function $z = f_{(x,y)}$ is a parametric surface $\vec r_{(x,y)} = \langle x, y, f_{(x,y)}\rangle$

$\Rightarrow \frac{\partial \vec r}{\partial x} = \langle 1, 0, \frac{\partial f}{\partial x}\rangle, \quad \frac{\partial \vec r}{\partial y} = \langle 0, 1, \frac{\partial f}{\partial y}\rangle$

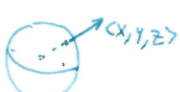
$\Rightarrow \frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y\end{vmatrix} = \langle -f_x, -f_y, 1\rangle \Rightarrow \text{magnitude} = \sqrt{f_x^2+f_y^2+1}$.

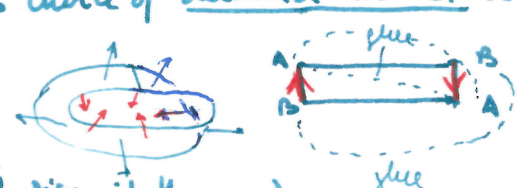
Conclusion: $\displaystyle\iint_S g_{(x,y,z)}\, dS = \iint_R g(x,y,f_{(x,y)})\sqrt{1+f_x^2+f_y^2}\, dA.$

Here R is the domain of f, need not be a rectangle.

§4 Surface integrals of vector fields:

Need to work with orientable surfaces = continuous choice of outward normal vector

Eg: sphere 

Non-eg: Möbius strip 

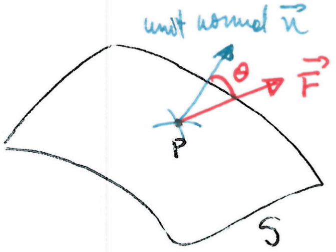
take outward normal $= \frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v}$. (or w/ opposite sign if the orientation of S is reversed)

Main example: flux integrals = find net flow of the vector field across the surface

Def $\text{Flux}(\vec F = \langle f, g, h\rangle) = \iint_S \vec F\cdot\vec n\, dS$ (same as in $\mathbb{R}^2$)

$\vec n$ → unit outer normal

Why?



$\vec{F}\cdot\vec{n} = |F|\cos\theta$

- $\theta = 0 \Rightarrow$ comp of F in the $\vec{n}$-direction is $\vec{F}\cdot\vec{n} = |\vec{F}|$
  all of $\vec{F}$ flows across S in the direction of $\vec{n}$
- $\theta = \pi \Rightarrow$ All of $\vec{F}$ flows across S ——— opposite to $\vec{n}$.
  $[\,\vec{F}\cdot\vec{n} = -|\vec{F}|\,]$
- $\theta = \frac{\pi}{2} \Rightarrow \vec{F}\cdot\vec{n} = 0$ no flow through S at P.

$\Rightarrow$ Flux just adds up these components of $\vec{F}$ normal to the surface at all points of S

Q: How to compute the flux?

$\vec{n}$ is unit vector w/ direction $\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}$ $\Rightarrow$ $\boxed{\vec{n} = \dfrac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}}$

(we assume this direction is compatible w/ the orientation of S)

Conclusion:

$$\text{Flux}(\vec{F}) = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R F\cdot\frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\cancel{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}}\,\cancel{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}\, dA$$

$$= \iint_R F(\vec{r}(u,v))\cdot\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right) dA$$

Special case: S = graph of $f(x,y) \Rightarrow \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} = \langle -f_x, -f_y, 1\rangle$

If $\vec{F}_{(x,y,z)} = \langle g, h, p\rangle_{(x,y,z)}$ Then,

$$\text{Flux}(\vec{F}) = \iint_R \left(-g_{(x,y,f(x,y))} f_{x(x,y)} - h_{(x,y,f(x,y))} f_{y(x,y)} + p_{(x,y,f(x,y))}\right) dA$$