Lecture XXXIV : §.15.7 Stokes' Theorem

Recall: Surface S in $\mathbb{R}^3$ parameterized by $\vec{r} : R \longrightarrow \mathbb{R}^3$

$$\vec{r}_{(u,v)} = < x(u,v), y(u,v), z(u,v) >$$

Typically: $R = \underset{u}{[a,b]} \times \underset{v}{[c,d]}$ rectangle.

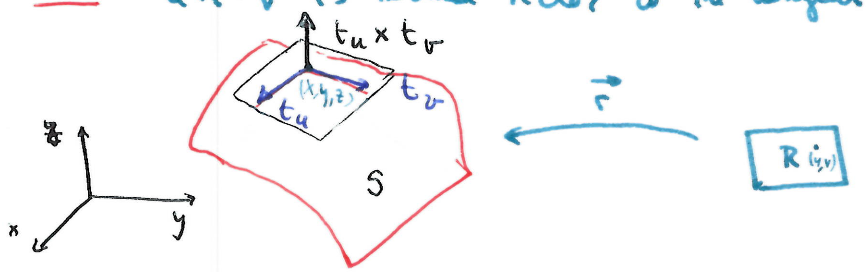
Def: Given $f = f_{(x,y,z)} : S \longrightarrow \mathbb{R}$ cont. defined $\iint_S f_{(x,y,z)}\, dS = \iint_R f_{(x_{(u,v)}, y_{(u,v)}, z_{(u,v)})} \cdot |t_u \times t_v| d$

where $t_u = \frac{\partial \vec{r}}{\partial u} = < \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >_{(u,v)}$

$t_v = \frac{\partial \vec{r}}{\partial v} = < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} >_{(u,v)}$

& $(t_u \times t_v)_{(u,v)}$ is not the $\vec{0}$ vector field

Note: $t_u \times t_v$ is normal vector to the tangent plane to S at a point $\vec{r}_{(u,v)}$ in S.



- **Special case:** Graphs of functions $z = g(x,y)$ are parametric surfaces $\vec{r}_{(x,y)} = < x, y, g_{(x,y)} >$

$\Rightarrow \frac{\partial \vec{r}}{\partial x} = < 1, 0, \frac{\partial g}{\partial x} >$, $\frac{\partial \vec{r}}{\partial y} = < 0, 1, \frac{\partial g}{\partial y} >$ $\boxed{R = \text{domain of } g.}$ not nec. a rectangle

$\rightarrow \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = < -g_x, -g_y, 1 > \Rightarrow$ magnitude $= \sqrt{g_x^2 + g_y^2 + 1}$

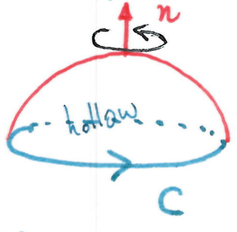
↳ usual normal to tangent plane to the graph of g!

Conclusion: $\iint_S f_{(x,y,z)}\, dS = \iint_R f(x, y, g_{(x,y)}) \sqrt{1 + g_x^2 + g_y^2}\, dA$

§1 Surface integrals of vector fields:

Orientation of a surface S = continuous choice of outward (unit) normal vector to S

If the surface is bounded by a simple & closed curve C we can use the right-hand rule to define the unique direction of the outward normal



rotate counterclockwise around $\vec{n}$

Recall: $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is a normal vector, but depending on the orientation of S the vector could have opposite dir to $\vec{n}$, so $\boxed{\vec{n} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}}$.

$\vec{F} = \langle f, g, h \rangle$ v. field in S, want to find the net flow of the vector field across S

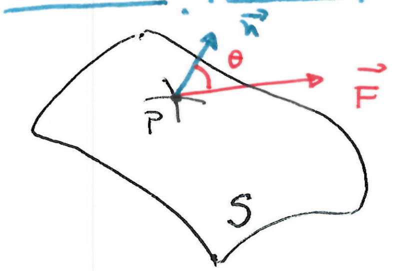
Def: $\text{Flux}(\vec{F}) = \iint_S \underbrace{\vec{F} \cdot \vec{n}}_{\text{outer unit normal}} \, dS = \iint_R F_{(x(u,v), y(u,v), z(u,v))} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) dA$

(if orientation of S gives $\vec{n} = \dfrac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}$

- Special case: S = graph of a function $z = P(x,y)$ $\Rightarrow$ $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle -P_x, -P_y, 1 \rangle$

$$\text{Flux}(\vec{F}) = \iint_R \left(-f_{(x,y,P(x,y))} P_{x(x,y)} - g_{(x,y,P(x,y))} P_{y(x,z)} + h_{(x,y,P(x,y))} \right) dA$$

- Why we have this formula for flux?



$\vec{F} \cdot \vec{n} = |F| \cos \theta$

- If $\theta = 0$ $\Rightarrow$ comp of $\vec{F}$ in the $\vec{n}$-direction is $\vec{F} \cdot \vec{n} = |\vec{F}|$ so all of $\vec{F}$ flows across S in the dir. of $\vec{n}$.
- If $\theta = \pi$ $\Rightarrow$ ———————— $-\vec{n}$ because $\vec{F} \cdot \vec{n} = -|\vec{F}|$.
- If $\theta = \frac{\pi}{2}$ $\Rightarrow$ $\vec{F} \cdot \vec{n} = 0$ so no flow through S at P.

Flux just adds up the components of $\vec{F}$ normal to the surface at all pts of S.

§2 Curl & Green's Thm:

Def: $F = \langle f, g, h \rangle$ v. field with all 1st order partials cont. The curl of F is a v. field defined as

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) - j \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) + k \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)}_{\text{curl in } \mathbb{R}^2.}$$

Each comp. gives the rotation/orientation of $\vec{F}$ in the corresp. coord. planes.

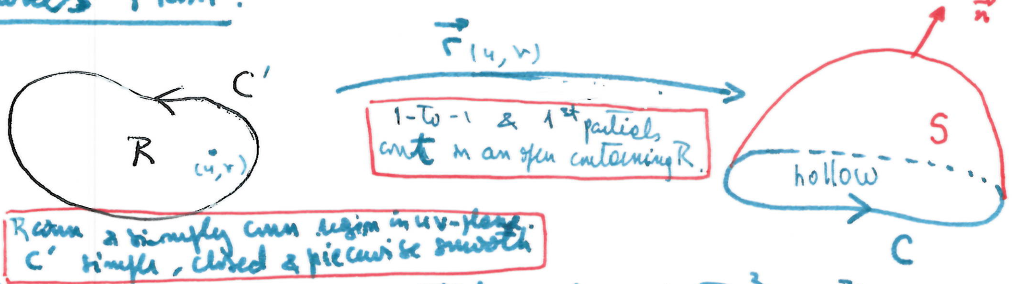
{ i-comp$(\nabla \times F) > 0$ $\Leftrightarrow$ F is rotating counterclockwise in xy-plane (ie the v. field projected to this plane)
& similar for the other 2 components.

Green's Theorem: $\vec{F} = \langle f, g \rangle$ w/ cont 1st partials on a region R, conn & simply conn in $\mathbb{R}^2$, enclosed by a simple & closed curve C, oriented counterclock., then

$$\oint_C F \cdot dr = \iint_R \underbrace{g_x - f_y}_{= \text{curl}(F)} \, dA.$$

GOAL: Extend this result to param. surfaces in $\mathbb{R}^3$. ⟶ STOKES' THM.

§3 Stokes' Thm:



1-To-1 & 1st partials cont on an open containing R.

R a conn simply conn region in uv-plane.
C' simple, closed & piecewise smooth

orientation of C is inherited from C' through the param $\vec r$

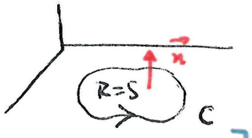
Stokes' Theorem: Fix an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed bound curve C whose orientation is consistent with that of S. Assume $\vec F$ is a v. field with continuous first partials on S. Then:

$$\text{Circ}(F \text{ in } C) = \oint_C F\cdot d\gamma = \iint_S \underset{\substack{\downarrow\\ \text{curl}(F)}}{(\nabla\times F)}\cdot\vec n\; dS = \text{Flux}(\nabla\times\vec F \text{ in } S)$$

where $\vec n$ is the unit normal to S determined by the orientation of S & it's comp w/ orient of C.
& $\gamma(t)=\langle x(t), y(t), z(t)\rangle$ parameterizes C

Note: If S lies in the x,y plane, we take $\vec r(x,y)=\langle x,y,0\rangle$ & $\vec n=\langle 0,0,1\rangle$ if C is counterclockwise oriented. Then curl(F) = curl(<f, g, h>) = curl(<f, g, 0>)

$$\& \quad \text{Circ}(F \text{ in } C) \underset{\substack{\downarrow\\ \boxed{\text{Green's Thm}}}}{=} \iint_R g_x - f_y \; dA = \iint_R (\nabla\times\vec F)\cdot\vec n\; dS = \text{Flux}(\nabla\times\vec F \text{ in } S).$$



compatible orientations

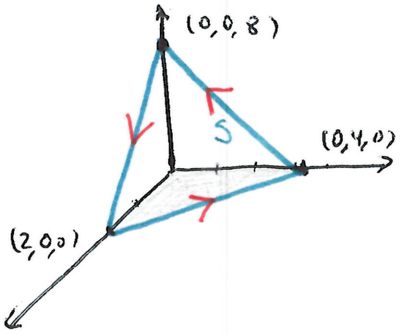
So Stokes' Theorem generalizes Green's Theorem.

Corollary: [curl $\vec F = 0$ implies $\vec F$ conservative on simply conn regions in $\mathbb{R}^3$.]

Suppose that $\nabla\times F=\vec 0$ (so $f_y = g_x$, $f_z = h_x$, $g_z = h_y$) on an open, connected & simply conn region D of $\mathbb{R}^3$. Then $\oint_C F\cdot dr = 0$ on all closed simple sm curves C in D & so $\vec F$ is conservative.

Pf/ given C in D as above, we can find a smooth oriented surface S in D bounded by C. Then Stokes' Thm gives $\oint_C \vec F\cdot dr = \iint_S \underset{\overset{\shortparallel}{0}}{(\nabla\times\vec F)}\cdot\vec n\; dS = 0$ ✓.

Examples ① $\vec F = \langle z, -z, x^2-y^2\rangle$ & C = three line segments bounding the plane $z = 8-4x-2y$ in the first octant oriented clockwise when looking from (0,0,0)



Find the circulation of $\vec F$.

- Soln 1: Parameterize 3 segments & use def of Circ(F). counter clockwise from above!
- Soln 2: Use Stoke's Theorem.

Notice: S is the graph of $z = f(x,y) = 8-4x-2y$ defined on R = 

Orientation of S ? $\vec{n} \overset{?}{=} \dfrac{\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}}{\left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right|}$ or opposite orientation ?

$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = < -f_x, -f_y, 1 > = < 4, 2, \underbrace{1}_{} >$ & compatible w/ boundary orientation ✓

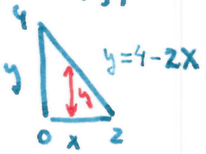
↳ points outwards

so $\oint_C F \cdot dr = \iint_S \underbrace{(\nabla \times \vec{F}) \cdot \vec{n}}_{\text{scalar function}} \, dS = \iint_R <1-2y, 1-2x, 0> \cdot <4, 2, 1> \, dA = \iint_R 6-4x-8y\,d$

$= \int_0^2 \int_0^{4-2x} 6-4x-8y \, dy\,dx = \boxed{\frac{-88}{3}}$

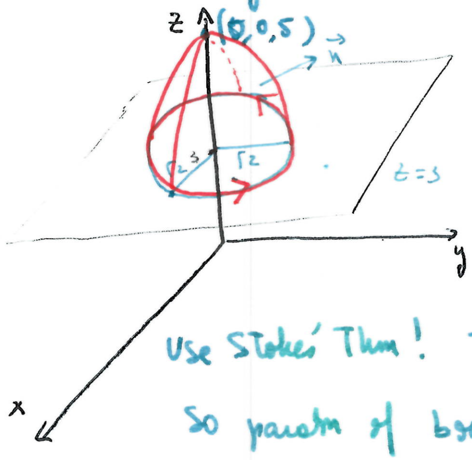
$\nabla \times \vec{F} = \begin{vmatrix} i & j & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2-y^2 \end{vmatrix} = < -2y+1, -(2x-1); 0-0 > = < 1-2y, 1-2x, 0$

$\vec{r}(x,y) = < x, y, 8-4x-2y >$



② Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$ where $\vec{F} = < -xz, yz, xye^z >$ & S = cap of the parabolo

$z = 5 - x^2 - y^2$ above the plane $z = 3$ & $\vec{n}$ points upwards.



Paraboloid intersected w/ $z=3$ gives $3 = 5 - x^2 - y^2$ $\boxed{x^2+y^2 = 2}$

vertex = (0,0,5)

Again: $\vec{r}(x,y): \{x^2+y^2 \le 2\} \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (x, y, \overbrace{5-x^2-y^2}^{= f(x,y)})$

Use Stokes' Thm! The normal direction $\vec{n}\uparrow$ induces counterclockwise orientation for C

So param of boundary: $\vec{r}(t) = < x, y, z > = < \sqrt{2}\cos t, \sqrt{2}\sin t, 3 >$ $0 \le t \le 2\pi$

$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS \underset{\text{Stokes}}{=} \oint_C \vec{F} \cdot dr = \int_0^{2\pi} < -\sqrt{2}\cos t \, 3, \sqrt{2}\sin t \, 3, 2\sin t \cos t \, e^3 > \cdot < -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 > dt$

$= \int_0^{2\pi} \underbrace{6\cos t \sin t + 6\cos t \sin t + 0}_{= 12\cos t \sin t} \, dt = 6 \sin^2 t \Big|_{t=0}^{t=2\pi} = \boxed{0}$

- Without Stokes' Theorem: $\vec{n} \overset{?}{=} \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = < -f_x, -f_y, 1 > = < +2x, 2y, 1 >$ upwards!

$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{x^2+y^2 \le 2} < xe^{5-x^2-y^2} - y, -x - ye^{5-x^2-y^2}, 0 > \cdot < 2x, 2y, 1 > dA = \iint_{x^2+y^2 \le 2} x^2 e^{5-x^2-y^2} - 2xy - 2xy - y^2 e^{5-x^2-y^2} \, dA =$

$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xz & yz & xye^z \end{vmatrix} = < xe^z - y, -(ye^z + x), 0 > \Rightarrow \nabla \times \vec{F}_{(\vec{r}(x,y))} = < xe^{5-x^2-y^2} - y, -x - ye^{5-x^2-y^2}, 0 >$

Use polar coordinates! $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\begin{matrix} 0 \le \rho \le \sqrt{2} \\ 0 \le \theta \le 2\pi \end{matrix}$

So $$\iint_{x^2+y^2\le 2} (x^2-y^2)e^{5-x^2-y^2} - 4xy \, dA = \int_0^{\sqrt{2}}\int_0^{2\pi} \left(\rho^2(\cos^2\theta - \sin^2\theta)e^{5-\rho^2} - 4\rho^2\cos\theta\sin\theta\right)\rho \, d\theta \, d\rho$$

$$= \int_0^{\sqrt{2}}\int_0^{2\pi} \rho^3 e^{5-\rho^2} \underbrace{(\cos^2\theta - \sin^2\theta)}_{=\cos 2\theta} - 4\rho^3 \underbrace{\cos\theta\sin\theta}_{=\frac{\sin 2\theta}{2}} \, d\theta \, d\rho$$

$$= \int_0^{\sqrt{2}} \rho^3 e^{5-\rho^2} \frac{\sin 2\theta}{2}\Bigg|_{\theta=0}^{\theta=2\pi} + 2\rho^3 \frac{\cos 2\theta}{2}\Bigg|_{\theta=0}^{\theta=2\pi} d\rho = \int_0^{\sqrt{2}} 0 \, d\rho = \boxed{0}$$

§4 Proof of Stoke's Thm:

$\vec{F} = \langle f, g, h \rangle$ Assume $\vec{n}$ & $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ have the same direction.

$\gamma(t) = \langle x(t), y(t), z(t) \rangle$ $a \le t \le b$ param of the curve C

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

We start by writing the identity we wish to prove:

$$\mathrm{Circ}(F,C) \oint_C F \cdot d\gamma = \int_a^b f(x(t),y(t),z(t)) \cdot x'(t) + g(x(t),y(t),z(t))\, y'(t) + h(x(t),y(t),z(t))\, z'(t) \, dt$$

$$= \oint_C f \, dx + \oint_C g \, dy + \oint_C h \, dz.$$

$$\mathrm{Flux}(\nabla\times F, S) = \iint_S (\nabla \times F) \cdot \vec{n} \, dS = \iint_S \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle \cdot \vec{n} \, dS$$

$$t_u \times t_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \langle y_u z_v - y_v z_u, -(x_u z_v - z_u x_v), x_u y_v - x_v y_u \rangle$$

So $$\mathrm{Flux}(\nabla\times F, S) = \iint_R \{(h_y - g_z)(y_u z_v - y_v z_u) - (f_z - h_x)(x_u z_v - z_u x_v)$$
$$+ (g_x - f_y)(x_u y_v - x_v y_u)\} dA$$

We prove the Theorem by showing the following 3 identities holds

$$\begin{cases} \oint_C f \, dx = \iint_R -(f_z(x_u z_v - z_u x_v) + f_y(x_u y_v - x_v y_u)) dA \\ \oint_C g \, dy = \iint_R g_x(x_u y_v - x_v y_u) - g_z(y_u z_v - y_v z_u) dA \\ \oint_C h \, dz = \iint_R h_x(x_u z_v - z_u x_v) + h_y(y_u z_v - y_v z_u) dA \end{cases}$$

We write the explicit calculation for the first identity (the others are proven similarly)

The (RHS) can be expressed as:

$$-f_z(x_u z_v - z_u x_v) - f_y(x_u y_v - x_v y_u) = \frac{\partial}{\partial u}\left(f(x(u,v), y(u,v), z(u,v)) \frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(f(x(u,v), y(u,v), z(u,v)) \cdot \frac{\partial x}{\partial u}\right)$$

Reasons:
- $x_{vu} = x_{uv}$ because the function $\vec{r}$ was sufficiently nice & smooth.
- Chain rule applied to $\tilde{f}(u,v) := f(x(u,v), y(u,v), z(u,v))$. & product rule

Using Green's Theorem for R & the boundary curve C', we get.

$$\iint_R \frac{\partial}{\partial u}\left(\tilde{f}(u,v)\, x_v\right) - \frac{\partial}{\partial v}\left(\tilde{f}(u,v)\, x_u\right) dA = \oint_{C'} \tilde{f}(u,v)\, x_u\, du + \tilde{f}(u,v)\, x_v\, dv$$

$$\downarrow$$

$$\overset{\text{v.field}}{\underset{\text{in uv-plane}}{=}} \langle \tilde{f}(u,v) x_u, \tilde{f}(u,v) x_v \rangle$$

We can take a parametrization of C': $\alpha(t) = \langle u(t), v(t) \rangle$, $a \le t \le b$ & then $\gamma(t)$ can be chosen as

$$\vec{\gamma}(t) = \vec{r}(\alpha(t)) = \langle x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)) \rangle$$

We conclude

$$\oint_{C'} \tilde{f}(u,v)\, x_u\, du + \tilde{f}(u,v)\, x_v\, dv = \oint_C f\, dx.$$

$$\| $$

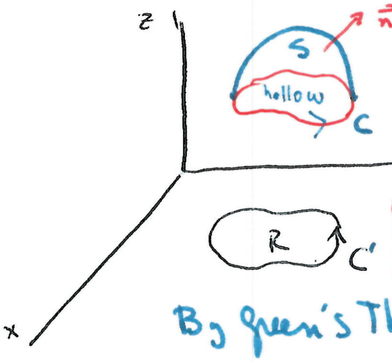
$$\int_a^b f(\vec{r}(\alpha(t)))\, dx$$

$\square$

Proof in 1 special case

Assume S is the graph of a function $P(x,y)$ so $\vec{r}(x,y) = \langle x, y, P(x,y) \rangle$
P has 2nd order partials that are cont.

$z = P(x,y)$
$dz = P_x dx + P_y dy$



$$\vec{n} = \frac{\langle -P_x, -P_y, 1 \rangle}{|\langle -P_x, -P_y, 1 \rangle|}$$

$$\oint_C F \cdot dr = \oint_{C'} f\, dx + g\, dy + h\, dz = \oint_{C'} f\, dx + g\, dy + h\,(P_x dx + P_y dy)$$

$$= \oint_{C'} (\underbrace{f + hP_x}_{=:M(x,y)})\, dx + (\underbrace{g + hP_y}_{=:N(x,y)})\, dy.$$

By Green's Thm this $= \iint_R (N_x - M_y)\, dA.$

By the chain rule $M_y = (f(x,y,P(x,y)) + h(x,y,P(x,y))P_x)_y = f_y + f_z \cdot P_y + hP_{xy} + P_x(h_y + h_z P_y)$

& similarly $N_x = g_x + g_z P_x + hP_{yx} + P_y(h_x + h_z P_x)$

Since $P_{xy} = P_{yx}$ by hypothesis, $\iint_R (N_x - M_y)\, dA = \iint_R -P_x(-g_z + h_x) - P_y(-h_x + f_z) + (g_x - f_y)\, dA =$

$= \iint_S (\nabla \times \vec{F}) \vec{n}\, dS$ as we wanted.