Lecture XXXV : §15.5–15.7 Rotational V. fields, curls & Stokes' Theorem

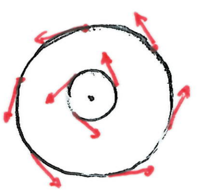
§1 Curl of a general rotational vector field:

Special vector fields: $F_{(x,y,z)} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is a constant vector, $\vec{a} \neq \vec{0}$, & $\vec{r} = \langle x, y, z \rangle$. In coordinates:

$$\vec{F} = \vec{a} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x \rangle$$

Def: $\vec{F}$ as above is called a general rotational field in $\mathbb{R}^3$.

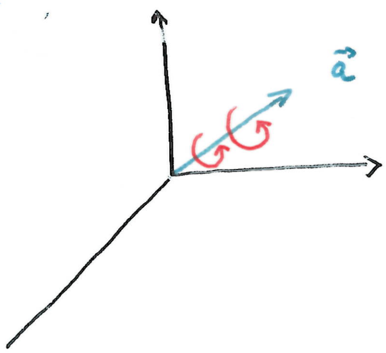
Eg: $a_1 = a_2 = 0$, $a_3 = 1$ we get $\vec{F} = \langle -y, x, 0 \rangle = \langle -y, x \rangle$ usual rotational vector field in $\mathbb{R}^2$



concentric circles: $|\vec{F}|$ is constant & $\vec{F}$ is tangent to the circles.

axis of rotation: positive z-axis. curl $= g_x - f_y = 1 - (-1) = 2 > 0$ & counterclockwise rotation looking from above.

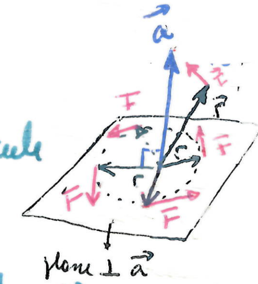
In general: $\vec{F} = \vec{a} \times \vec{r}$ is a rotational v. field with axis of rotation in the direction of $\vec{a}$



3 Properties ① $\underset{\operatorname{div}(\vec{F})}{\nabla \cdot \vec{F}} = \underbrace{\frac{\partial}{\partial x}(a_2 z - a_3 y)}_{=0} + 0 + 0 = 0$, so $\boxed{\operatorname{div} \vec{F} = 0}$

② $\vec{F} = \vec{a} \times \vec{r}$ circles around the vector $\vec{a}$ in the counterclockwise direction looking along $\vec{a}$ from head to tail.

Why? $\vec{F}$ lies in an orthogonal plane to $\vec{a}$ & use right-hand rule



③ $\operatorname{curl}(F) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i(a_1 - (-a_1)) - j(-a_2 - a_2) + k(a_3 - (-a_3))$
$= \langle 2a_1, 2a_2, 2a_3 \rangle = 2\vec{a}$.

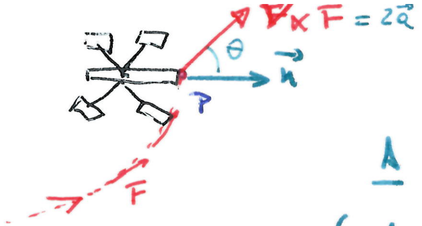
$$\begin{cases} f = a_2 z - a_3 y \\ g = a_3 x - a_1 z \\ h = a_1 y - a_2 x \end{cases}$$

So curl$(\vec{F})$ has the same direction as the axis of rotation of $\vec{F}$ ($\vec{a}$).

Furthermore $|\operatorname{curl}(\vec{F})| = |\nabla \times \vec{F}| = 2|\vec{a}|$

Here, $|\vec{a}|$ is the constant angular speed of rotation = rate at which a particle in the v. field rotates about the axis of rotation of the v. field $\vec{F}$.

Remark Suppose a paddle wheel is placed in $\vec{F}$ at a pt P with the axis of the wheel in the direction of a unit vector $\vec{n}$. Q: For which $\vec{n}$'s does the paddle spin faster?



$$\mathrm{comp}_{\vec n}\, \nabla\times\vec F = \frac{(\nabla\times\vec F)\cdot\vec n}{\underbrace{|\vec n|}_{=1}} = |\nabla\times\vec F|\cdot\underbrace{|\vec n|}_{1}\cos\theta$$

Want to maximize this! So $\cos\theta = 1$, ie $\theta = 0$

Conclusion: Fastest rotation speed when $\vec n$ has the same dir as $\nabla\times\vec F$.

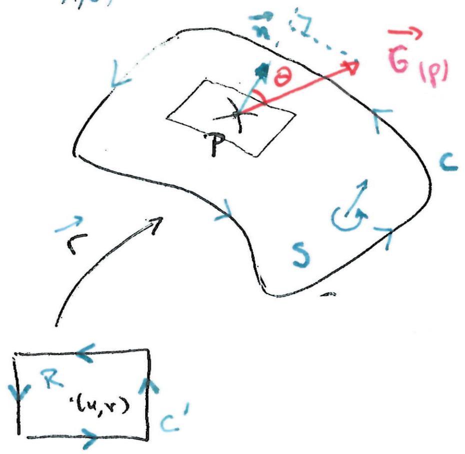
• If $\theta = \pm\frac{\pi}{2}$, then no rotation at all (the wheel doesn't spin).

§2 Interpreting the curl via Stokes' Thm:

Write C = curve bounding the surface S

Flux integral along a surface = net flow of the vector field across the surface (eg: amount of water flowing through a pipe's cross section).

If $\vec G_{(x,y,z)} = \langle f, g, h\rangle_{(x,y,z)}$ is a vector field & P is a pt on a surface S in $\mathbb{R}^3$ w/ unit outer normal $\vec n$ perpendicular to the tangent plane to S at P.



Flux at P is the scalar component of $\vec G_{(P)}$ in the direction of $\vec n$.

$$\mathrm{comp}_{\vec n}\, \vec G_{(P)} = \frac{\vec G_{(P)}\cdot\vec n}{|\vec n|} = |\vec G_{(P)}|\,\underbrace{|\vec n|}_{=1}\cos\theta = |\vec F_{(P)}|\cos\theta$$

• If $\theta = 0$ $\Rightarrow$ all of $\vec G_{(P)}$ flows across S in the dir. of $\vec n$.

• If $\theta = \pi$ $\Rightarrow$ ________ opposite to $\vec n$

• If $\theta = \frac{\pi}{2}$ $\Rightarrow \vec G\cdot\vec n = 0$ no flow through S at P.

$\Rightarrow$ Flux integral adds up these scalar components to compute the net flow of $\vec G$ across S.

• Assume the unit outer normal to S has the same dir as $t_u \times t_v = \frac{\partial\vec r}{\partial u}\times\frac{\partial\vec r}{\partial v}$.

Take any vector field $\vec F$, then $\vec G = \mathrm{curl}(\vec F) = \nabla\times\vec F$ is a new vector field.

Stokes' Thm: Assume $\vec r$ is 1-to-1 in the interior of R & with cont 1st partials in an open set in the uv-plane containing R. Assume R is conn & simply connected & bounded by a simple closed piecewise smooth curve C'. Then

$$\mathrm{Circ}(F, C) = \oint_C F\cdot dr = \iint_S (\nabla\times\vec F)\cdot\underbrace{\vec n}_{\text{unit outer normal}}\, dS = \mathrm{Flux}(\nabla\times\vec F, S)$$

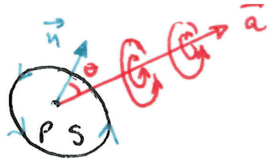
($\vec r$ is param of C)

• We can use this to define the average flux of $\nabla\times\vec F$ over S:

Def Average flux of $\nabla\times\vec F$ over S $= \frac{1}{\mathrm{area}(S)}\oint_C F\cdot dr = \frac{1}{\mathrm{area}(S)}\iint_S (\nabla\times\vec F)\cdot\vec n\, dS$

Special case: $\vec F = \vec a\times\vec r$ general rot. field in $\mathbb{R}^3$.

Know: $\nabla\times\vec F = 2\vec a$.



Take S = small circular disk centered at a point P w/ normal $\vec{n}$ forming an angle θ with $\vec{a}$.

Take C = boundary circle, counterclockwise oriented from above.

$$\boxed{\text{Average flux}} = \frac{1}{\text{area}(S)} \iint_S \underbrace{\underbrace{(\nabla \times \vec{F})}_{\overset{\shortparallel}{2\vec{a}}} \cdot \vec{n}}_{\text{constant} = 2|\vec{a}|\cos\theta} \, dS = \frac{1}{\text{area}(S)} (\nabla \times \vec{F}) \cdot \vec{n} \underbrace{\iint_S 1 \, dS}_{\overset{\shortparallel}{\text{area}(S)}} = \boxed{2|a|\cos\theta}$$

- If $\theta = 0 \Rightarrow$ average circ has max value $2|\vec{a}| = 2$ angular speed.
- If $\theta = \frac{\pi}{2} \Rightarrow$ " " is 0.

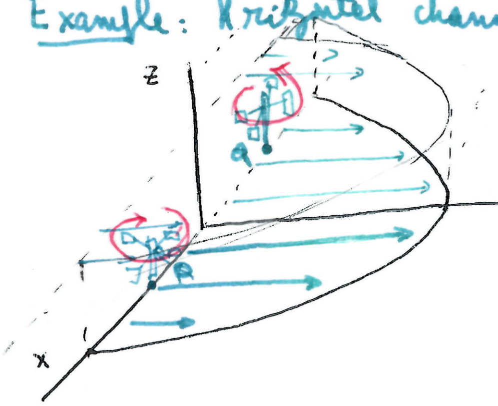
Conclusions:
- The scalar comp of $\nabla \times \vec{F}$ at P in the dir of $\vec{n}$ = average flux of $\nabla \times \vec{F}$ on S.
- dir$(\nabla \times \vec{F})$ at P is the direction that maximizes the average flux of curl of F on S

Equiv, we should orient the axis of a paddle in this direction to maximize the angular speed.

- General v. fields: given P, take S_R small disk centered at a pt P w/ radius R & call C_R = boundary of S_R

$$\lim_{R \to 0} (\text{Average flux}(F, S_R)) = \lim_{R \to 0^+} \frac{1}{\text{area}(S_R)} \iint_{S_R} \underbrace{(\nabla \times \vec{F}) \cdot \vec{n}}_{\text{closed to } \nabla \times \vec{F}_{(P)} \cdot \vec{n} \text{ if } R \text{ is small}} \, dS = (\nabla \times \vec{F})_{(P)} \cdot \vec{n} \quad \text{fixed number.}$$

Example: Horizontal channel flow $\vec{v} = \langle 0, 1-x^2, 0 \rangle$ velocity v. field $|x|, |z| \leq 1$.



Place paddle wheels at $P = (\frac{1}{2}, 0, 0)$ & $Q = (-\frac{1}{2}, 0, 0)$.

Q: Find axis for each paddle wheel that makes it spin fastest.

- If in x-axis $\Rightarrow$ NO spin! (flow strikes the upper & lower halves of the wheel symmetrically!).
- If in y-axis, no spin! (flow parallel to axis!).
- If in z-axis: ① flow for $x < \frac{1}{2}$ greater than for $x > \frac{1}{2}$ so CLOCKWISE SPIN at P.

② flow for $x < -\frac{1}{2}$ is smaller than for $x > -\frac{1}{2}$ so COUNTERCLOCKWISE SPIN at Q

Curl = ? $\nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 1-x^2 & 0 \end{vmatrix} = \langle 0, 0, -2x \rangle$ has z-axis direction.
$\Rightarrow$ max spin in this dir!

$\vec{n} = k = \langle 0, 0, 1 \rangle$

$(\nabla \times \vec{v}) \cdot \vec{n} = -2x > 0$ at Q $\Rightarrow$ counterclock rot!

$(\nabla \times \vec{v}) \cdot \vec{n} = -2x < 0$ at P $\Rightarrow$ clock rot!

§3 Consequences of Stokes' Theorem: $\vec{F} = \langle f, g, h \rangle$

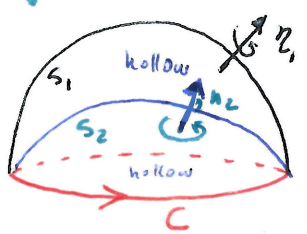
Corollary 1: If $f_y = g_x$, $f_z = h_x$ & $g_z = h_y$ on an open, connected & simply conn. region D of $\mathbb{R}^3$, then $\oint_C F \cdot dr = 0$ on all closed simple sm. curves C in D & so $\vec{F}$ is conservative.

Proof: Given C as above, we can find a smooth oriented surface S in D bounded by C. Note that $\nabla \times \vec{F} = \vec{0}$ by hypothesis, so Stoke's Theorem gives

$$\oint_C \vec{F} \cdot dr = \iint_S \underbrace{(\nabla \times \vec{F})}_{=\vec{0}} \cdot \vec{n} \, dS = 0 \checkmark.$$

Extensions: A curve C can bound more than 1 surface & we can use Stokes' Th to simplify the calculation of $\mathrm{Circ}(F, C)$ or $\mathrm{Flux}(\nabla \times F, S_1)$ using $\mathrm{Flux}(\nabla \times F, S_2)$

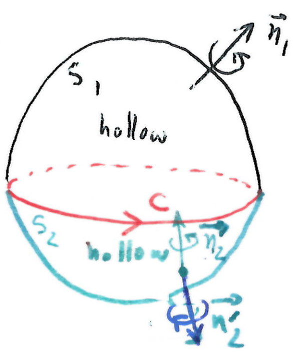
CASE 1:



$\vec{n}_1$ & $\vec{n}_2$ both point upwards.

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot n_1 \, dS = \oint_C F \cdot dr = \iint_{S_2} \nabla \times \vec{F} \cdot n_2 \, dS$$

CASE 2:



$\vec{n}_1$ & $\vec{n}_2'$ point in opposite directions & $\vec{n}_2'$ is NOT comp. with the orientation of C. But $\vec{n}_1$ & $\vec{n}_2'$ are comp. w/ orientation of S, $\vec{n}$.

$$\iint_{S_1} \nabla \times \vec{F} \cdot \vec{n}_1 \, dS = \oint_C \vec{F} \cdot dr$$

$$\iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}_2' \, dS = \oint_{C^{op}} \vec{F} \cdot dr = -\oint_C \vec{F} \cdot dr = -\iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}_2 \, dS$$

S_1 & S_2 form a closed surface S w/ common boundary curve C

$$\text{So} \iint_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \iint_{S_2} \nabla \times \vec{F} \cdot \underbrace{\vec{n}}_{=\vec{n}_2'} \, dS + \iint_{S_1} \nabla \times \vec{F} \cdot \underbrace{\vec{n}}_{=\vec{n}_1} \, dS = -\oint_C \vec{F} \cdot dr + \oint_C \vec{F} \, dr = \boxed{0}$$

Conclusion: On any closed, smooth oriented surface S, enclosing a region in $\mathbb{R}^3$, $\iint_C \nabla \times \vec{F} \cdot \vec{n} \, dS = 0$.