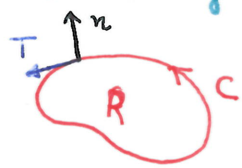
Lecture XXXVI: §15.8 Divergence Theorem

Recall: 2 versions of Green's Theorem:



- C simple & closed, piecewise sm. COUNTERCLOCKWISE ORIENTED
- R conn & simply conn with boundary = C

① Circulation form $\vec{F} = \langle f, g \rangle$, $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\oint_C F \cdot dr = \iint_R \operatorname{curl}(F)\, dA \qquad (\operatorname{curl} = g_x - f_y)$$

② Flux form:

$$\int_C F \cdot \vec{n}\, ds = \iint_R \operatorname{div}(F)\, dA \qquad (\operatorname{div}(F) = f_x + g_y)$$

(ds: arc length; $\vec{n}$: unit outward normal)

generalized: • $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$

$F = \langle f, g, h \rangle$ • $\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = f_x + g_y + h_z$

Stokes Theorem generalizes V.① $\qquad \oint_C F \cdot dr = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$

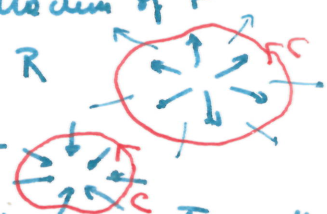
C curve in $\mathbb{R}^3$ bounding a surface S in $\mathbb{R}^3$

TODAY: Generalize (V2) = DIVERGENCE THM. Compute the flux of $\vec{F}$ across a surface S by a triple integral of its divergence over a region D bounded by S.

Recall: divergence in $\mathbb{R}^2$ measures net expansion or contraction of F

$\operatorname{div}(F) > 0$ on R $\Rightarrow$ we have a source for F in R

$\operatorname{div}(F) < 0$ on R $\Rightarrow$ ——— sink ———



The integral on the right-hand side of ② measures the cumulative effect of sinks or sources

§1. Divergence Theorem:

Theorem: Fix D a conn & simply connected region in $\mathbb{R}^3$ bounded by a smooth oriented surface S. Let $\vec{F}$ be a vector field in D whose components have continuous 1st partials

Then: $\operatorname{Flux}(F,S) = \iint_S F \cdot \vec{n}\, dS = \iiint_D \operatorname{div}(F)\, dV,$

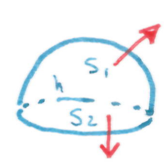
where $\vec{n}$ is the unit outward normal vector on S (pointing outside D!)

Note: (RHS) is the cumulative expansion or contraction of $\vec{F}$ over D.

Example Last time $\vec{F} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$ general rotational field.

$a_1 = a_3 = 1 \quad a_2 = 0 \Rightarrow \vec{F} = \langle -y, x - z, y \rangle$

$S =$ northern hemisphere of $x^2 + y^2 + z^2 = a^2$ (S_1) + base in the xy-plane (S_2)



We can compute the flux as $\text{Flux}(\vec{F}, S) = \text{Flux}(\vec{F}, S_1) + \text{Flux}(\vec{F}, S_2)$

S_1 parametrized by $\vec{r}_1(x,y) = \langle x, y, \underbrace{\sqrt{a^2-x^2-y^2}}_{=f(x)} \rangle$ $\quad x^2+y^2 \leq a^2$.

$\frac{\partial \vec{r}_1}{\partial x} \times \frac{\partial \vec{r}_1}{\partial y} = \langle -f_x, -f_y, 1 \rangle \to$ points upwards ✓

$$\iint_{S_1} \vec{F} \cdot \vec{n}\, dS = \iint_{x^2+y^2 \leq a^2} \langle -y, x - \sqrt{a^2-x^2-y^2}, y \rangle \cdot \langle \frac{x}{\sqrt{\ }}, \frac{y}{\sqrt{\ }}, 1 \rangle\, dA \ \ldots\ldots$$

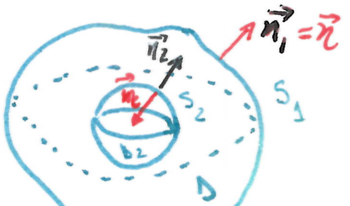
$$\iint_{S_2} \vec{F} \cdot \underset{\langle 0,0,-1 \rangle}{\vec{n}}\, dS = \iint_{x^2+y^2 \leq a^2} \langle -y, x, y \rangle \cdot \langle 0,0,-1 \rangle\, dA = \iint_{x^2+y^2 \leq a^2} -y\, dA$$

<u>EASIER</u>: Use Divergence Thm! $\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$ (true for gen. rot fields!)

So $\text{Flux}(F, S) = \iiint_D 0\, dV = \boxed{0}$.

§ 2 <u>Extensions to other regions</u>: (like we did for Green's Theorem)

<u>GOAL</u>: Use the Theorem for hollow regions (with holes ~~so~~ not simply connected!)



$D_2 \cup D = D_1$

- D bounded by 2 surfaces S_1 & S_2. $[S_1 \cup S_2 = S]$
- How to extend this result?

(1) S_1 is bounding a solid D_1 = conn & simply connected & has outer normal $\vec{n}_1 = \vec{n}$ (normal for D_1; normal for D)

(2) S_2 is bounding a solid D_2 : conn & simply conn with outer normal $= \vec{n}_2 = -\vec{n}$

$$\text{Flux}(F, S_1) = \iiint_{D_1} \nabla \cdot \vec{F}\, dV \quad \& \quad \text{Flux}(\vec{F}, S_2) = \iiint_{D_2} \nabla \cdot \vec{F}\, dV$$

$$\text{Flux}(F, S_1) = \iint_{S_1} F \cdot \vec{n}_1\, dS = \iint_{S_1} F \cdot \vec{n}\, dS$$

$$\text{Flux}(\vec{F}, S_2) = \iint_{S_2} F \cdot \vec{n}_2\, dS = \iint_{S_2} F \cdot (-\vec{n})\, dS = -\iint_{S_2} F \cdot \vec{n}\, dS$$

<u>Conclusion</u>: $\text{Flux}(\vec{F}, S) = \iint_S F \cdot \vec{n}\, dS = \iint_{S_1} F \cdot \vec{n}\, dS + \iint_{S_2} F \cdot \vec{n}\, dS$

$$= \iint_{S_1} F \cdot \vec{n}_1\, dS - \iint_{S_2} F \cdot \vec{n}_2\, dS = \iiint_{D_1} \nabla \cdot \vec{F}\, dV - \iiint_{D_2} \nabla \cdot \vec{F}\, dV = \iiint_D \nabla \cdot \vec{F}\, dV$$

<u>KEY</u>: Use the correct directions to the normal vectors to each component of S.

Application: $\vec F$ may no be differentiable at a pt (say $(0,0,0)$) & we can isolate it with D_2 = small ball centered at this point.

Example: $\vec F = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} = \dfrac{\vec r}{|\vec r|^3}$. Find the outward flux across

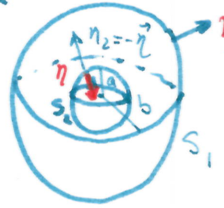
(a) S_1 bounding $D = \{(x,y,z) : a^2 \le x^2+y^2+z^2 \le b^2\}$

(b) S_2 = any sphere enclosing $(0,0,0)$

v. field on $\mathbb{R}^3 \setminus \{(0,0,0)\}$

Soln: Recall $\mathrm{div}(\vec F) = 0$ (in general $\mathrm{div}\left(\dfrac{\vec r}{|\vec r|^p}\right) = \dfrac{3-p}{|\vec r|^p}$)

(a) Use extended Div Thm:



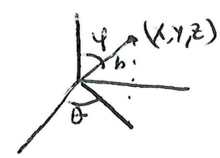
$$\mathrm{Flux}(F,S) = \iiint_{D} 0 \, dV = 0$$

D = $a^2 \le x^2+y^2+z^2 \le b^2$

(b) We can take the inner sphere with an arbitrary center, and (a) still holds:

$$\text{Now: } 0 = \iiint_D 0 \, dV = \underbrace{\iint \vec F \cdot \vec \eta_1 \, dS}_{\text{outer sph } S_1} - \underbrace{\iint \vec F \cdot \vec \eta_2 \, dS}_{\text{inner sphere } S_2}$$

so inner sphere flux = outer sphere S_1 w/ radius b.



We use spherical coordinates $\vec r(\varphi,\theta) = \langle b \sin\varphi \cos\theta, b \sin\varphi \sin\theta, b\cos\varphi \rangle$

$$\left| \frac{\partial \vec r}{\partial \varphi} \times \frac{\partial \vec r}{\partial \theta} \right| = \left| \begin{vmatrix} i & j & k \\ b\cos\varphi\cos\theta & b\cos\varphi\sin\theta & -b\sin\varphi \\ b\sin\varphi(-\sin\theta) & b\sin\varphi\cos\theta & 0 \end{vmatrix} \right| = \underbrace{|\mathrm{Jac}(\varphi,\theta)|}_{\text{sph to cartesian}} = b^2 \sin\varphi .$$

also $\vec\eta_1 = \dfrac{\langle x,y,z\rangle}{b}$

so $\vec F \cdot \vec \eta_1 = \dfrac{|\langle x,y,z\rangle|^2}{|b|^3 \cdot b} = \dfrac{b^2}{b^4} = \dfrac{1}{b^2}$

$$\iint_{S_1} F\cdot\vec\eta_1 \, dS = \int_0^{2\pi}\int_0^{\pi} b^{-2} \cdot b^2 \sin\varphi \, d\varphi \, d\theta = 2\pi(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi} = -2\pi(-1-1) = \boxed{4\pi}$$

Application 2: Electric field due to a point charge Q at $(0,0,0)$ is $\vec E(x,y,z) = \dfrac{Q}{4\pi\varepsilon_0} \dfrac{r}{|r|^3}$

ε_0 = permittivity of free space (physical constant).

GAUSS' LAW: If S is a surface enclosing a point charge Q, the flux of the electric field across S is

$$\iint_S E \cdot \vec\eta \, dS = \frac{Q}{4\pi\varepsilon_0} \cdot 4\pi = \boxed{\frac{Q}{\varepsilon_0}}$$

Replace charged centered at a point w/ a charge density $q(x,y,z)$ (charge per unit volume) then $Q = \iiint_D q(x,y,z) \, dV$ D is region enclosed by S.

$$\iint_S E\cdot\vec\eta \, dS = \frac{1}{\varepsilon_0}\iiint_D q(x,y,z) \, dV$$

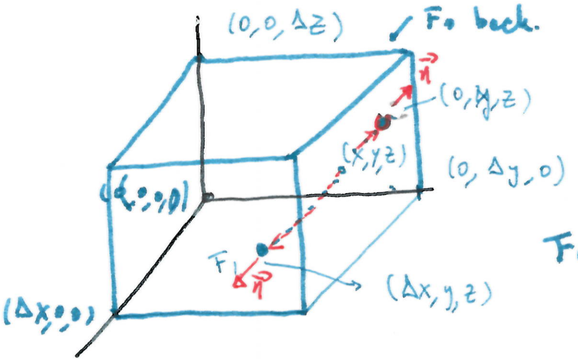
§3 Interpreting the divergence theorem using mass transport

$\begin{cases} \vec{v} = \text{velocity field of a material (e.g. water)} \\ \rho = \text{constant density of the material} \end{cases}$ $\Rightarrow$ $\vec{F} = \rho\vec{v} = \langle f, g, h\rangle$ is the MASS TRANSPORT (flux density) with units $\frac{\text{mass}}{\text{vol}} \cdot \frac{\text{length}}{\text{time}} = \frac{\text{mass}}{\text{area}\cdot\text{time}}$.

$\vec{F}(P)$ = mass of material flowing past the pt (P) in each coordinate direction per unit of area per unit of time.

[Area$(S)\, \vec{F}\cdot\vec{n}$ gives $\approx$ flux$(\vec{F}, S)$ when S is small]

Take D = cube with faces parallel to the coordinate planes & vertex $(0,0,0)$ & edges of lengths $\Delta x, \Delta y, \Delta z$.



- Pick (x,y,z) in D

GOAL: Compute flux of material across the faces of D.

Faces: $x=0$ (F_0) & $x=\Delta x$ (F_1) $\qquad$ Area$(F_0) = \Delta y \cdot \Delta z$ = Area(F_1)

$$\text{Flux}(\vec{F}, F_0) + \text{Flux}(\vec{F}, F_1) \cong \vec{F}(\Delta x, y, z)\cdot \underbrace{\vec{n}}_{\langle 1,0,0\rangle} \Delta y \Delta z + \vec{F}(0,y,z)\cdot \underbrace{\vec{n}}_{\langle -1,0,0\rangle} \Delta y \Delta z.$$

$$= \big(f(\Delta x, y, z) - f(0,y,z)\big)\, \Delta y \Delta z.$$

$$= \frac{f(\Delta x, y, z) - f(0,y,z)}{\Delta x}\, \Delta V \;(\text{vol}(D)) \approx \frac{\partial f}{\partial x}(0,y,z)\,\Delta V$$

Similarly: $\text{Flux}(\vec{F}, y=0 \cup y=\Delta y) \approx \frac{g(x,\Delta y,z) - g(x,0,z)}{\Delta y}\,\Delta V \approx \frac{\partial g}{\partial y}(x,0,z)\,\Delta V$

$\text{Flux}(\vec{F}, z=0 \cup z=\Delta z) \approx \frac{h(x,y,\Delta z) - h(x,y,0)}{\Delta z}\,\Delta V \approx \frac{\partial h}{\partial z}(x,y,0)\,\Delta V$

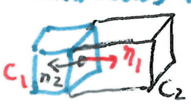
Conclusion: Add contributions of all 6 faces:

net flux out of the cube $\underset{\substack{\Delta x, \Delta y, \Delta z \\ \text{small} \\ \text{so } x,y,z \to (0,0,0)}}{\approx} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)\Big|_{(0,0,0)} \Delta V = \nabla\cdot F_{(0,0,0)} \cdot \Delta V.$

Note: We can replace $(0,0,0)$ by any pt in the region.

If we break D into N small cubes then net flux out of $D \approx \sum_{k=1}^{N} \nabla\cdot F_{(x_k^*, y_k^*, z_k^*)} \cdot \Delta V_k$ $\quad (x_k^*, y_k^*, z_k^*)$ in Cube$_k$.

Contributions of faces shared by 2 cubes cancel OUT! Take limit as $\Delta x, \Delta y, \Delta z \to 0$.



$\vec{n}_2 = -\vec{n}_1$

$$\boxed{\iint F\cdot\vec{n}\, dS} \underset{\text{DEF}}{=} \text{Net flux out of } D = \boxed{\iiint_D \nabla\cdot F\, dV} \Rightarrow \text{Divergence Thm!}$$