Practice Midterm 1 (Solutions)

Problem 1: (a) $\vec{r}(t) = \langle |t-2|, 0\rangle$ for t in $\mathbb{R}$

The 1-variable function $f(u) = |u|$ is continuous on $\mathbb{R}$ but not differentiable at $u=0$ $\left(\lim_{u\to 0^+} \frac{f(u)-0}{u} = 1 \quad , \lim_{u\to 0^-} \frac{f(u)-0}{u} = -1 \right)$, so $\vec{r}$ is not differentiable at $t=2$.

(b) As we saw in class the circle with center $(0,0)$ & radius a has curvature $\kappa = \frac{1}{a}$.

Computation:

$$\vec{r}(t) = \langle a\cos t, a\sin t\rangle \qquad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle \Rightarrow |\vec{r}\,'(t)| = a$$

so $T(t) = \langle -\sin t, \cos t\rangle \rightsquigarrow \kappa(t) = \frac{|T'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{a}$

$$T'(t) = \langle -\cos t, -\sin t\rangle$$

Alternative: view the circle in $\mathbb{R}^3$ (in xy-plane):

$$\vec{r}(t) = \langle a\cos t, a\sin t, 0\rangle$$

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t, 0\rangle \rightsquigarrow |\vec{r}\,'(t)| = a$$

$$\vec{r}\,''(t) = \langle -a\cos t, -a\sin t, 0\rangle$$

so $\kappa(t) = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{a^2}{a^3}\cdot|\vec{k}| = \frac{1}{a}$ (as we showed above ✓)

$$\vec{r}\,'\times\vec{r}\,'' = a^2\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0\end{vmatrix} = a^2\left(0\vec{i} - 0\vec{j} + \underbrace{(\sin^2 t + \cos^2 t)}_{=1}\vec{k}\right) = a^2\vec{k}$$

(c) Two planes are parallel if their normal directions are parallel $\Rightarrow$ only need to change the constant term in the equations

Original plane: $2x - 5y - z = 0$

2 parallel planes: $2x-5y-z = 1$, $2x-5y-z=2$.

(d) $u = \langle u_1, u_2\rangle$ where $u_1\cos\frac{\pi}{5} + u_2\sin\frac{\pi}{5} = 0$.

All examples are of the form $u = a\langle \sin\frac{\pi}{5}, -\cos\frac{\pi}{5}\rangle$ for a in $\mathbb{R}$.

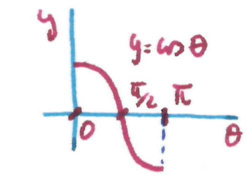
In particular: $u = \vec{0}$, $u = \langle \sin\frac{\pi}{5}, -\cos\frac{\pi}{5}\rangle$ satisfy the requirement.

Problem 2 : (a) $2u - v = \langle 6,4,4\rangle - \langle 1,-4,6\rangle = \langle 5,8,-2\rangle$

(b) We compute the angle θ with the formula $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$$\begin{cases} \theta \text{ is acute} \iff \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} > 0 \iff \vec{u}\cdot\vec{v} > 0 \\ \theta \text{ is right} \iff \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = 0 \iff \vec{u}\cdot\vec{v} = 0 \\ \theta \text{ is obtuse} \iff \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} < 0 \iff \vec{u}\cdot\vec{v} < 0 \end{cases}$$

why?

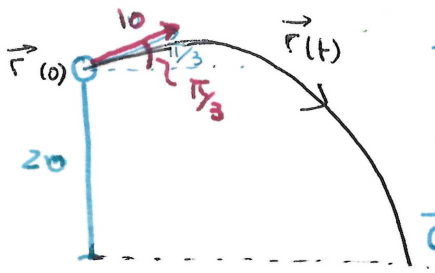


So $\vec{u}\cdot\vec{v} = 3\cdot 1 + 2(-4) + 2\cdot 6 = 7 > 0$, so the angle is ACUTE.

(c) We use the formula $\text{proj}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \dfrac{7}{17}\langle 3,2,2\rangle = \boxed{\left\langle \frac{21}{17}, \frac{14}{17}, \frac{14}{17}\right\rangle}$

$|\vec{u}| = \sqrt{9+4+4} = \sqrt{17}$

Problem 3 :



$\vec{r}(0) = \langle 0, 20\rangle$

$\vec{v}(0) = 10\langle \cos\frac{\pi}{3}, \sen\frac{\pi}{3}\rangle = 10\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$\vec{a}(t) = \langle 0, -g\rangle$ (gravitational force is the only one acting)

$\boxed{g = -9.8}$

We integrate twice to compute $\vec{r}(t)$:

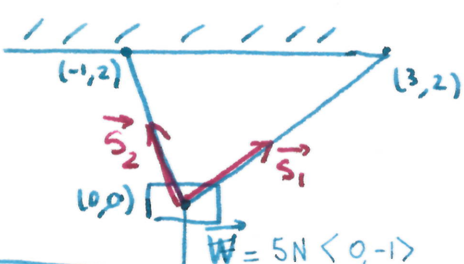
$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0,-g\rangle\,dt = \langle 0, -gt\rangle + \vec{C}$

$\langle 5, 5\sqrt{3}\rangle = \vec{v}(0) = \vec{C} \implies \vec{v}(t) = \langle 5, 5\sqrt{3} - gt\rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 5, 5\sqrt{3} - gt\rangle\,dt = \langle 5t, 5\sqrt{3}t - g\frac{t^2}{2}\rangle + \vec{C}_2$

$\langle 0, 20\rangle = \vec{r}(0) = \vec{C}_2 \implies \boxed{\vec{r}(t) = \langle 5t, 20 + 5\sqrt{3}t - g\frac{t^2}{2}\rangle}$

Problem 4 :



The weight induces an action in the (unit) direction $\langle 0,-1\rangle$ of magnitude 5N, so $\boxed{\vec{W} = \langle 0,-5\rangle}$

The system is in equilibrium, which means $\boxed{\vec{S}_1 + \vec{S}_2 + \vec{W} = \vec{0}}$ $(*)$

We know the direction of the forces acting on each string: $\text{dir}(\vec{S}_1) = \langle 3,2\rangle - \langle 0,0\rangle = \langle 3,2\rangle$

$\text{dir}(\vec{S}_2) = \langle -1,2\rangle - \langle 0,0\rangle = \langle -1,2\rangle$

So the unit directions are

$$u.\mathrm{dir}(\vec{S}_1) = \frac{\langle 3,2\rangle}{\sqrt{13}}, \qquad u\,\mathrm{dir}(\vec{S}_2) = \frac{\langle -1,2\rangle}{\sqrt{5}}.$$

It remains to compute the magnitude of ~~these~~ two vectors. Fix $a = |\vec{S}_1|$, $b = |\vec{S}_2|$

We compute a & b from $(*)$:

x-comp: $\frac{3a}{\sqrt{13}} - \frac{b}{\sqrt{5}} = 0 \implies b = \frac{3\sqrt{5}\,a}{\sqrt{13}}$

y-comp: $\frac{2a}{\sqrt{13}} + \frac{2b}{\sqrt{5}} = 5$ ⤳ substitute b expression here $5 = \frac{2a}{\sqrt{13}} + \frac{6\sqrt{5}\,a}{\sqrt{13}\sqrt{5}} = \frac{8a}{\sqrt{13}}$

so $a = \frac{5\sqrt{13}}{8}$ & $b = \frac{15\sqrt{5}}{8}$. Then $\vec{S}_1 = \frac{5}{8}\langle 3,2\rangle = \boxed{\langle \frac{15}{8}, \frac{5}{4}\rangle}$ ✓

$\vec{S}_2 = \frac{15}{8}\langle -1,2\rangle = \boxed{\langle -\frac{15}{8}, \frac{15}{4}\rangle}$

(Check: $\vec{S}_1 + \vec{S}_2 + \vec{W} = \langle 0,0\rangle$ ✓)

Problem 5: We use the formula for curves in $\mathbb{R}^2$:

- $\vec{r}(t) = \langle 2\cos t, 3\sin t\rangle$

$\vec{r}'(t) = \langle -2\sin t, 3\cos t\rangle \implies |\vec{r}'(t)| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4\sin^2 t + 4\cos^2 t + 5\cos^2 t}$ ($9 = 4+5$, $= 4$)

$= \sqrt{4 + 5\cos^2 t}$

$$\vec{T}(t) = \frac{\langle -2\sin t, 3\cos t\rangle}{\sqrt{4+5\cos^2 t}}$$

$$\vec{T}'(t) = \frac{\langle -2\cos t, -3\sin t\rangle}{\sqrt{4+5\cos^2 t}} + \frac{10\cos t\sin t}{2}\frac{\langle -2\sin t, 3\cos t\rangle}{(\sqrt{4+5\cos^2 t})^3}$$

$$= \frac{(4+5\cos^2 t)\langle -2\cos t, -3\sin t\rangle + 5\cos t\sin t\,\langle -2\sin t, 3\cos t\rangle}{(\sqrt{4+5\cos^2 t})^3}$$

$$= \frac{\langle -8\cos t - 10\cos^3 t - 10\cos t\sin^2 t, -12\sin t - 15\sin t\cos^2 t + 15\cos^2 t\sin t\rangle}{(\sqrt{4+5\cos^2 t})^3}$$

$$= \frac{\langle \cos t(-8 - 10(\cos^2 t + \sin^2 t)), -12\sin t\rangle}{(\sqrt{4+5\cos^2 t})^3}$$

$$= \frac{\langle -18\cos t, -12\sin t\rangle}{(\sqrt{4+5\cos^2 t})^3}$$

So $$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{6\sqrt{9\cos^2 t + 4\sin^2 t}}{(\sqrt{4+5\cos^2 t})^4} = \frac{6}{(\sqrt{4+5\cos^2 t})^3}$$

- The maximum value of $\kappa(t)$ is achieved when the denom is minimal:

$$\left(\sqrt{4+5\cos^2 t}\right)^3 \text{ is minimum} \iff 4+5\cos^2 t \text{ is minimum}$$

$$\iff \cos^2 t = 0, \text{ so } t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\boxed{\text{Max } \kappa = 6/8}$$

- The minimum value of $\kappa(t)$ is attained when the denominator is maximal:

$$\left(\sqrt{4+5\cos^2 t}\right)^3 \text{ is maximal} \iff 4+5\cos^2 t \text{ is maximal}$$

$$\iff \cos^2 t = 1 \text{ so } t = 0, \pi, 2\pi.$$

$$\boxed{\text{Min } \kappa = \frac{6}{(\sqrt{9})^3} = \frac{6}{27}}$$

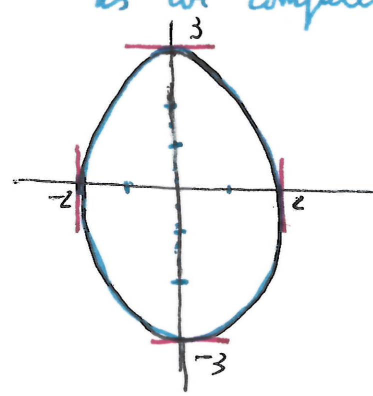
- Alternative: we can view the curve in $\mathbb{R}^3$ as $\vec{r}(t) = \langle 2\cos t, 3\sin t, 0\rangle$ and use the alternative formula for $\kappa(t)$. $\vec{r}\,'(t) = \langle -2\sin t, 3\cos t, 0\rangle$

$$\vec{r}\,''(t) = \langle -2\cos t, -3\sin t, 0\rangle \quad , \quad |\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4+5\cos^2 t}.$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \begin{vmatrix} i & j & k \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + (-6\sin^2 t - 6\cos^2 t)\vec{k} = -6\vec{k}.$$

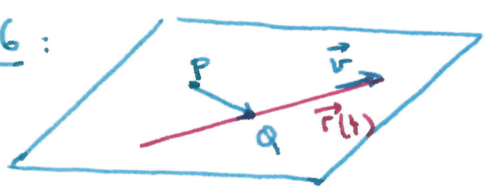
So $$\kappa(t) = \frac{|-6|}{\left(\sqrt{4+5\cos^2 t}\right)^3} = \frac{6}{\sqrt{4+5\cos^2 t}}$$ as we computed earlier.

- Remark: The curve is an ellipse



The curvature attains its maximum value at $(0,3)$ & $(0,-3)$ & its min. value at $(2,0)$ & $(-2,0)$

Problem 6:



We check if the point belongs to the line

$$\langle 2, \underline{-3}, 1\rangle \stackrel{?}{=} \langle 3+t, \underline{2}, 1-t\rangle$$

Answer: no! (The y-components will never be equal).

To determine the plane we need 2 directions:

- direction 1: direction of $\vec{r}(t) = \vec{v} = \langle 1, 0, -1\rangle$
- direction 2: $|\vec{PQ}|$ for any Q in the line, eg $Q = (3,2,1)$

$$|\vec{PQ}| = \langle 3-2, 2-(-3), 1-1\rangle = \langle 1, 5, 0\rangle$$

Using these 2 directions we build the normal direction as their cross product

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 5 & 0 \end{vmatrix} = 5\vec{i} - \vec{j}(1) + 5\vec{k} = \langle 5,-1,5\rangle$$

(Check: $\vec{n}\cdot\langle 1,0,-1\rangle = 0$ ✓ , $\vec{n}\cdot\langle 1,5,0\rangle = 0$ ✓)

Equation: $\vec{n}\cdot\langle x,y,z\rangle = \vec{n}\cdot\vec{OP} = \langle 5,-1,5\rangle\cdot\langle 2,-3,1\rangle$
$= 10+3+5 = 18$

$$\boxed{5x - y + 5z = 18}$$

(Check: P satisfies the eqn ✓
$\vec{r}(t)$ " " " for every t $\quad 18 \overset{?}{=} 5(3+t) - 2 + 5(1-t) = 15-2+5 = 18$ ✓)

Problem 7: For this, we must compute $\vec{a}(t)$, $\vec{T}(t)$ & $\vec{N}(t)$.

$$\vec{r}\,'(t) = \langle 2t, 1\rangle \to |\vec{r}\,'(t)| = \sqrt{4t^2+1}$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 2,0\rangle$$

$$\vec{T}(t) = \frac{\langle 2t,1\rangle}{\sqrt{4t^2+1}} \to \vec{T}'(t) = \frac{\langle 2,0\rangle}{\sqrt{4t^2+1}} - \frac{(8t)\langle 2t,1\rangle}{2(\sqrt{4t^2+1})^3}$$

$$= \frac{\langle 2(4t^2+1) - 8t^2, -4t\rangle}{(\sqrt{4t^2+1})^3} = \frac{\langle 2,-4t\rangle}{(\sqrt{4t^2+1})^3}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\langle 2,-4t\rangle}{\sqrt{4+16t^2}} = \frac{\langle 1,-2t\rangle}{\sqrt{1+4t^2}}$$

so $a_T(t) = \vec{a}(t)\cdot\vec{T}(t) = \langle 2,0\rangle\cdot\frac{\langle 2t,1\rangle}{\sqrt{4t^2+1}} = \boxed{\frac{4t}{\sqrt{4t^2+1}}}$

$a_N(t) = \vec{a}(t)\cdot\vec{N}(t) = \langle 2,0\rangle\cdot\frac{\langle 1,-2t\rangle}{\sqrt{1+4t^2}} = \boxed{\frac{2}{\sqrt{1+4t^2}}}$

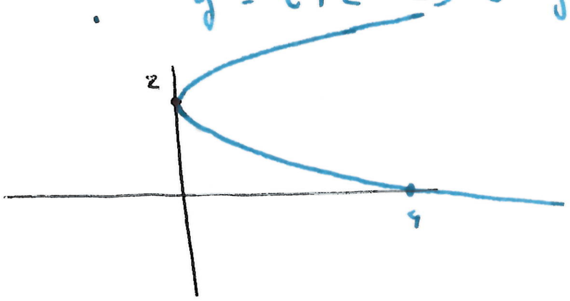
Alternative: We know $|\vec{N}(t)| = 1$, $\vec{N}(t)\cdot\vec{T}(t) = 0$, so from $\vec{T}(t)$ we ~~compute~~ know $\vec{N}(t) = \alpha\frac{\langle 1,-2t\rangle}{\sqrt{4t^2+1}}$ where $\alpha = \pm 1$.

If we draw $\vec{r}(t)$, we can decide the sign of α knowing that the vector $\vec{N}(t)$ points to the interior of the curve $\vec{N}$.

The equation of the curve is computed as follows.

$x = t^2$ $\Rightarrow$ $x = (y-2)^2$ so the curve is the graph of a parabola

$y = t+2 \Rightarrow t = y-2$



so the x-comp of $\vec{N}$ will always ~~point in~~ be positive

we can check at $t=0$, for example, to conclude $\boxed{\alpha = 1}$.

Problem 8: (i) The limit does NOT exist. We prove it by working with 2 paths and showing the limits along each of these paths ~~are~~ different (Path Test)

The domain of the function is $\mathbb{R}^2 \setminus \{(0,0)\}$ so any path through $(0,0)$ can be used!

If we look at the degrees in x and y of the numerator & denominator, we can guess which paths will work.

Along x-axis, $\vec{r}(x) = \langle x, 0\rangle$: $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x,y)\text{ on }x\text{-axis}}} \frac{6xy^2}{x^2+7y^4} = \lim_{x\to 0} 0 = \boxed{0}$

Write $x = x(t)$ as a polynomial in t where $x(0) = 0$

" $y = y(t)$ " " " in t where $y(0) = 0$.

Need to cancel the contributions between num & denom.

$$\deg_t(xy^2) = \deg_t x + 2\deg_t y$$

$$\deg_t(x^2) = 2\deg_t x, \quad \deg_t(y^4) = 4\deg_t y.$$

Want $\underbrace{\deg_t(x^2) = \deg_t(y^4)}_{=\deg(\text{denominator})} = \deg_t(xy^2)$ to get a limit $\neq 0$

[in general: $\deg(\text{denom}) = \min\{\deg_t(x^2), \deg_t(7y^4)\}$]

so $2\deg_t x = 4\deg_t y = \deg_t x + 2\deg_t y$

$$\Rightarrow \boxed{\deg_t x = 2\deg_t y}$$

Eg: $\begin{cases} y = t \\ x = mt^2 \end{cases}$ $\vec{r}_1(t) = \langle mt^2, t\rangle$ and $\vec{r}_1(t) \to \langle 0,0\rangle$ when $t\to 0$

[m = slope of the parabola, to be determined]

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\text{ along }\vec{r}_1(t)}} \frac{6xy^2}{x^2+7y^4} = \lim_{t\to 0} \frac{6(mt^2)t^2}{m^2t^4+7t^4} = \lim_{t\to 0}\frac{6m\cancel{t^4}}{\cancel{t^4}(m^2+7)} = \lim_{t\to 0}\frac{6m}{m^2+7} = \boxed{\frac{6m}{m^2+7}}$$

When $m=1$, lim is $6/8 \neq 0$.

Since we found two paths where the limits have different values, then we conclude the limit doesn't exist (The function fails the path test)

(ii) The limit does exist.

<u>Proof</u>: We notice that the function looks like $\frac{\sin(u)}{u}$ where $u = u(x,y) = x^2+4y^2$.

- The function $u(x,y)$ is continuous at $(0,0)$ & $u(0,0) = 0$.
- The function $g(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$ is continuous at $t = 0$

(Indeed, by L'Hôpital's Rule: $\underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{0/0 \text{ indeterminate}} = \lim_{t\to 0} \frac{(\sin t)'}{(t)'} = \lim_{t\to 0} \frac{\cos t}{1} = \cos 0 = \boxed{1}$

The function $f(x,y) = g(u(x,y))$. is continuous at $(0,0)$ by (*). In particular the limit exists & $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = g(u(0,0)) = g(0) = \boxed{1}$ ✓

$$(*):\quad \begin{tikzcd} \mathbb{R}^2 \arrow[r, "u(x,y)", "\text{cont}"'] \arrow[dr, "f(x,y) = g(u(x,y))"'] & \mathbb{R} \arrow[d, "g", "\text{cont}"'] \\ & \mathbb{R} \end{tikzcd}$$

Since g & u are continuous, then f is continuous.