Problem 1 \( (a) \) \( f(x, y) = \sin x \) : local max \( x = (2k+1) \frac{\pi}{2} \) \( \bar{\frac{\pi}{2}} \) \( \bar{\frac{\pi}{2}} + 2k\pi \). 

Other example: \( f(x, y) = \sin xy \).

Local max: \( xy = \frac{\pi}{2} + 2k\pi \) we have curve \( xy = \frac{\pi}{2} \) hyperbola.

Local max come from local max of the 1-variable function \( \sin (u) \).

(b) Since we know \( f \) is not differentiable at \((0,0)\) but we have a saddle point, we need to look for a function described by 2 formulas where the left & right partials are different.

Eq\(s\):

\[
\begin{align*}
\begin{cases}
 x, & x > 0 \\
 2x, & x < 0
\end{cases}
\end{align*}
\]

we see function has a saddle point at \( x = 0 \) because \( \lim_{x \to 0^+} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1 \) \( f_x \) does not exist at \((0,0)\).

We use a similar function:

\[ f(x, y) = |y| \]

\( f_x = 0 \)

\( f_y \) does not exist at \((0,0)\).

\[ f_y^+ (0,0) = 1 \]

\[ f_y^- (0,0) = -1 \]

We use our knowledge of gradients & level curves:

Asymptote (limit value = \(-\infty\))

\( A, B, C, D = \) local max
Problem 2: We write the velocity of the water:

\[ W(x,y) = \langle u(x,y), v(x,y) \rangle \]

The speed of the water is the magnitude of \( W(x,y) \), so

\[ S(x,y) = \sqrt{u^2 + v^2} \]

We write \( S(u,v) = \sqrt{u^2 + v^2} \) and note

\[
\begin{align*}
    u &= u(x,y) = x(1+x)(-1+2y) \\
    v &= v(x,y) = y(y-1)(-1+2x)
\end{align*}
\]

The rates of change of \( S \) are \( \frac{dS}{dx} \) and \( \frac{dS}{dy} \). We compute them using the Chain Rule:

\[
\left\{ \begin{array}{l}
\frac{dS}{dx} = Su \cdot u_x + Sv \cdot v_x \\
\frac{dS}{dy} = Su \cdot u_y + Sv \cdot v_y 
\end{array} \right.
\]

\[
S(x,y) = \left( 4x^4y^3 + 8x^3y^2 - 8x^2y + 12x^2y^2 - 8xy^3 + 8xy^2 + 2x^3 + 4y^3 
+ 3x^2 - 2y^4 + x - 8y^2y \right)^{1/2}
\]

\[
\frac{dS}{dx} = \frac{1}{S(x,y)} \left( (x(1+x))^2 (1+2y)(2y-1) + (1+2x)^2 (y(y-1)(-1+2x)) \right)
\]

\[
\frac{dS}{dy} = \frac{1}{S(x,y)} \left( (x(1+x))^2 (1+2y)(2) + (1+2x)^2 (y(y-1)(-1+2x)) \right)
\]

\[
\left( 4x^4y^3 + 8x^3y^2 - 8x^2y + 12x^2y^2 - 8xy^3 - 4x^3 + 8y^2y 
+ 3x^2 - 2y^4 + x - 8y^2y \right)^{1/2}
\]
Problem 3: (a) \( D_u(f) = \nabla f \cdot \vec{u} \)
\[
\nabla f = \langle f_x, f_y \rangle = \langle 9x^2y, 3x^2-1 \rangle \quad \Rightarrow \quad D_u(f) = \frac{1}{\sqrt{5}} \left( 18x^2y + 3x^2 - 1 \right).
\]
\[
\vec{u} = \frac{1}{\sqrt{5}} (-2, -1)
\]

(b) By a theorem discussed in class, the direction of maximum increase for \( f \) at \((1, 1, 1)\) is \( \nabla f(1, 1, 1) = \langle f_x(1, 1, 1), f_y(1, 1, 1), f_z(1, 1, 1) \rangle \)
\[
\nabla f = \langle yz^2, xe^2, 2xyz \rangle \quad \Rightarrow \quad \nabla f(1, 1, 1) = \langle 1, 1, 2 \rangle
\]
\[\therefore \text{since } \vec{u} \text{ must be a unit vector, then } \vec{u} = \frac{\langle 1, 1, 2 \rangle}{\sqrt{6}}
\]

(c) The normal direction is \( \langle -f_x(e, 1), -f_y(e, 1), 1 \rangle \)
\[
\vec{t} = \langle e, 1, 1 \rangle \text{ lies in the graph of } f \quad \Rightarrow \quad f(e, 1) = \frac{\ln 1}{e} = \frac{1}{e}
\]

(d) We use implicit differentiation. We assume \( z = z(x, y) \) locally near \((1, 0)\).

Take \( \frac{\partial}{\partial x} \) of the implicit equation: notice \( x \) and \( y \) are independent.

We get \( y + yz \frac{\partial z}{\partial x} + z + x \frac{\partial z}{\partial x} = 0 \) \( \text{(by the product rule)} \)
\[
\frac{\partial z}{\partial x} \left( 2yz + x \right) + z + y = 0
\]

\[\therefore \frac{\partial z}{\partial x} = \frac{-z-y}{x+2yz} \]

At the point \((1, 0)\), we know \( z = z(1, 0) \) can be
solved from substituting \( x = 1 \) & \( y = 0 \) in the implicit equation
\( 1 + 0 + 0.2^2 + 1.2 = 7 \quad \Rightarrow \quad z = 7 \)

Notice that \( \left( x + 2yz \right)_{(1, 0, 7)} = 1 \neq 0 \) so the expression \( \frac{\partial z}{\partial x} \) in

\[\text{(*) above is well-defined}
\]
\[
\frac{\partial z}{\partial x} \left( 1, 0 \right) = \frac{-7-0}{1} = -7
\]
(e) We use linear approximation: 
\[ z = f(x, y) = xy^2 - x^2 + y \]

\[ f(1, 1, 1.9) = f(1, 2) + f_x(1, 2)(1.1 - 1) + f_y(1, 2)(1.9 - 2) \]

\[ f_x = y^2 - 2x \Rightarrow f_x(1, 2) = 4 - 2 = 2 \]

\[ f_y = 2xy + 1 \Rightarrow f_y(1, 2) = 4 + 1 = 5. \]

\[ f(1, 2) = 4 - 1 + 2 = 5 \]

so \[ f(1, 1, 1.9) = 5 + 2(0.1) + 5(-0.1) = 5 - 3(0.1) = 4.7 \]

The change is \[ -0.3 \].

(f) We use the chain rule:

\[ z_t = 2x \cdot x_t + 2y \cdot y_t \]

\[ z = x^2 - 3y^2 + 20 \Rightarrow z_x = 2x \Rightarrow \int 2x(x + y) = 2. \]

\[ z_y = -6y \]

\[ x_t = -\cot t \Rightarrow x_t(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} \]

\[ y_t = \cot t \Rightarrow y_t(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \]

Conclusion: \[ \frac{\partial^2 z}{\partial t^2} \left( \frac{\pi}{4} \right) = \left( \frac{\sqrt{2}}{2} \right) \left( -\frac{\sqrt{2}}{2} \right) + \left( -6 \frac{\sqrt{2}}{2} \right) \frac{\sqrt{2}}{2} = \frac{-2}{2} - 3 \cdot \frac{2}{2} = -4 \]

Problem 4: We start by finding the critical points inside the ellipse.

Step 1: \( f \) is differentiable up to any order so the critical points are calculated as \( \nabla f(x, y) = 0 \).

\[ \nabla f = \langle f_x, f_y \rangle = \langle 1 - y, -x \rangle \]

so \( 1 - y = 0 \) & \( -x = 0 \) \( \Rightarrow x = 0 \)

so \( y = -1 \)

So only critical point is \( (0, 1) \) so it lies in the region.
We use the 2nd Derivative Test to decide the nature of the critical point.

\[ f_{xx} = (1-y)_x = 0 \]
\[ f_{xy} = (1-y)_y = 0 \]
\[ f_{yx} = (1-y)_y = -1 \]
\[ f_{yy} = (-x)_y = -1 \]

\[ \Rightarrow D(0,1) = 0 - (-1)(-1) = -1 < 0 \]

\( \Rightarrow (0,1) \) is a saddle point!

There are no other saddle points nor local max/min values in the interior of the ellipse.

**STEP 2.** We compute max/min values of \( f \) subject to the constraint

\[ g(x,y) = 9x^2 + y^2 = 9 \] (boundary of the ellipse)

We find these max/min values using Lagrange multipliers.

We must find \((x,y,\lambda)\) that solve 3 equations:

\[
\begin{align*}
    f_x &= \lambda g_x \\
    f_y &= \lambda g_y \\
    g(x,y) &= 9
\end{align*}
\]

From (2) we obtain: \( \lambda = \frac{-x}{2y} \) if \( y \neq 0 \)

Substitute in (1): \( 1-y = \frac{-x}{2y} (18x) \) so \( 2y(1-y) = -18x^2 \)

We replace in (3): \( 9x^2 + y^2 = -y(1-y) + y^2 = 2y^2 - y^2 = 9 \)

So \( 2y^2 = 9 \rightarrow y = \frac{1 \pm \sqrt{1+4 \cdot 2^2}}{2} = \frac{1 \pm \sqrt{17}}{2} \)

\[ x^2 = \frac{9-y^2}{9} = 1 - \frac{1}{9} \left( 173 \pm 2\sqrt{73} \right) = 9.16 - \frac{1}{9} (173 \pm 2\sqrt{73}) = 68.72 \pm \frac{2\sqrt{73}}{9.16} \]

\[ x = \frac{39 \pm \sqrt{73}}{72} \]

*STEP 3.* Compare values of \( f \) at these 2 points: \((a,b) = (9,5), (-9,5)\)

\[ a = \sqrt{\frac{39 + \sqrt{73}}{72}}, b = 1 - \frac{\sqrt{2}}{9} \]

\[ f(a,5) = a(1-b) = a(\frac{3 + \sqrt{73}}{9}) > 0 \Rightarrow \text{max value} \]

\[ f(-a,5) = -a(\frac{3 + \sqrt{73}}{9}) < 0 \Rightarrow \text{min value} \]

\( (0,0) \) was saddle so no need to compare it!
Problem 5: As usual, we start by drawing the picture of the solid.

\[ x^2 + y^2 = 1 \quad \text{(cylinder)} \]

\[ z = y \]

\[ z = 0 \]

We get

The projection is

\[ 0 \leq z \leq y \]

\[ 0 \leq x \leq \sqrt{1-x^2} \]

\[ 0 \leq y \leq \sqrt{1-x^2} \]

\[ 0 \leq x \leq 1 \]

\[ 0 \leq x \leq \frac{1}{2} \]

\[ \frac{1}{2} \leq x = \frac{1}{6} \]

\[ 0 \leq x \leq \frac{1}{3} \]

\[ 0 \leq x \leq \frac{3}{4} \]

\[ \int_{0}^{\frac{1}{3}} \left(1-x^2\right) \, dx = \int_{0}^{\frac{1}{6}} \left(\frac{1}{2} - 1\right) \, dx = \frac{1}{6} \]

In polar coordinates:

\[ R = \frac{r \cos \theta}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2} \]

\[ 0 \leq r \leq \sqrt{1-r^2} \]

\[ 0 \leq r \leq \sqrt{1-r^2} \]

\[ 0 \leq r \leq 0 \]

\[ 0 \leq r \leq 1 \]

\[ \frac{1}{3} \leq r = \frac{1}{6} \]

\[ \frac{1}{3} \leq r = \frac{1}{6} \]

\[ \frac{1}{3} \leq r = \frac{1}{2} \]

\[ \frac{1}{3} \leq r = \frac{1}{4} \]

Problem 6: (a) We start by drawing the region. We draw the cardioid by sampling some points \( \theta \in [0, 2\pi] \). Notice that the values at \( \theta \) and \( 2\pi - \theta \) will be the same.

For \( \theta \in \left[0, \frac{\pi}{2}\right] \) & \( \theta \in \left[\frac{3\pi}{2}, 2\pi\right] \) we have

\[ 0 \leq \omega \theta \leq 1 \]

But for \( \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \)

\[ r = 1 - \omega \theta \geq 1 \]

and \( r = 1 \) only for \( \theta = \frac{\pi}{2}, \frac{3\pi}{2} \).

We notice

\[ A_{\omega\mu}(R) = 2 A_{\omega\mu}(\triangle D) = 2 \left( A_{\omega\mu}(\triangle) + A_{\omega\mu}(D) \right) \]
\[ \text{Area} \left( D \right) = \text{Area} \left( \frac{1}{4} \text{ circle} \right) = \frac{1}{4} \left( \pi \left( 1 \right)^2 \right) = \frac{\pi}{4} \]

\text{Area} \left( D \right) \text{ can be computed in polar coordinates.}

\text{Region is defined as } \theta \in \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right], \quad 0 \leq r \leq 1 - \cos \theta

\text{Area} \left( D \right) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1 - \cos \theta} r \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{2} \bigg|_{r=0}^{r=1-\cos \theta} \, d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{(1 - \cos \theta)^2}{2} \, d\theta

= \int_{0}^{\frac{\pi}{2}} \frac{1 + \cos^2 \theta - 2 \cos \theta}{2} \, d\theta = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1 + \cos 2\theta}{4} - \cos \theta \, d\theta

\omega^2 \theta - \sin^2 \theta = \sin \theta (2\theta)

\omega^2 \theta - (1 - \omega^2 \theta) = 2 \omega^2 \theta - 1

= \frac{3}{4} \left[ -\cos \theta + \omega \sin \theta \right]_{0}^{\frac{\pi}{2}} = \frac{3\pi}{8} - 1

\text{(inclusion)} \quad \text{Area} \left( R \right) = 2 \left( \frac{15}{4} + \frac{3\pi}{8} - 1 \right) = 2 \left( \frac{5\pi}{8} - 1 \right) = \frac{5\pi}{4} - 2

(b) \text{ We draw the region:}

\[ y = 3x^2 \]

Find the intersection:

\[ 3x^2 = 16 - x^2 \]

\[ 4x^2 = 16 \]

\[ x^2 = 4 \quad \text{so} \quad x = \pm 2 \quad \text{and} \quad y = 16 - 4 = 12 \]

It's a type I region.

\[ \text{Area} = \int_{-2}^{2} 3x^2 \, dx = \int_{-2}^{2} -3x^2 + (16 - x^2) \, dx = \int_{-2}^{2} -4x^2 + 16 \, dx \]

\[ = \left[ -\frac{4}{3} x^3 + 16x \right]_{-2}^{2} \]

\[ = \left( \frac{4}{3} \cdot 8 + 16 \cdot 2 \right) - \left( -\frac{4}{3} \cdot 8 + 16 \cdot (-2) \right) = 2 \left( \frac{4}{3} \cdot 8 + 16 \cdot 2 \right) = 2 \left( \frac{32}{3} + 32 \right) \]

\[ = \frac{128}{3} \]