Practice Midterm 2 (Solutions)

Problem 1 (a) $f(x,y) = \sin x$: local max = $x = (2k+1)\frac{\pi}{2} = \frac{\pi}{2}, \frac{\pi}{2}+2\pi, \ldots$ $\boxed{\frac{\pi}{2}+2k\pi}$

Other example: $f(x,y) = \sin xy$:

Local max: $xy = \frac{\pi}{2} + 2k\pi$ we have curves $xy = \frac{\pi}{2}$ → hyperbola.

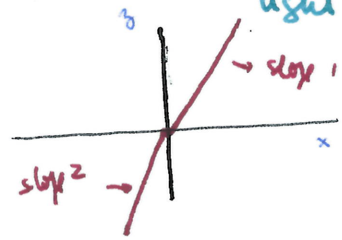
$xy = 2\pi + \frac{\pi}{2}$ ....

local max come from local max of the 1-variable function $\sin(u)$.

(b) Since we know f is not differentiable at $(0,0)$ but we have a saddle point, we need to look for a function described by 2 formulas where the left & right partials are different.

Eg:

$$f(x,y) = \begin{cases} x & x>0 \\ +2x & x<0 \end{cases}$$



We see the function has a saddle point at $x=0$ because $\begin{cases} 2x<0 & \text{if } x<0 \\ x>0 & \text{if } x>0 \\ f(0,y)=0 \end{cases}$

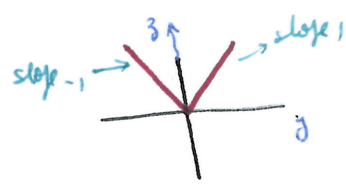
$\frac{\partial f}{\partial y} \equiv 0$. but $\lim\limits_{x\to 0^+} \frac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0^+} \frac{x}{x} = 1$

$\lim\limits_{x\to 0^-} \frac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0^-} \frac{2x}{x} = 2$

f_x does not exist at $(0,0)$

(c) We use a similar function:

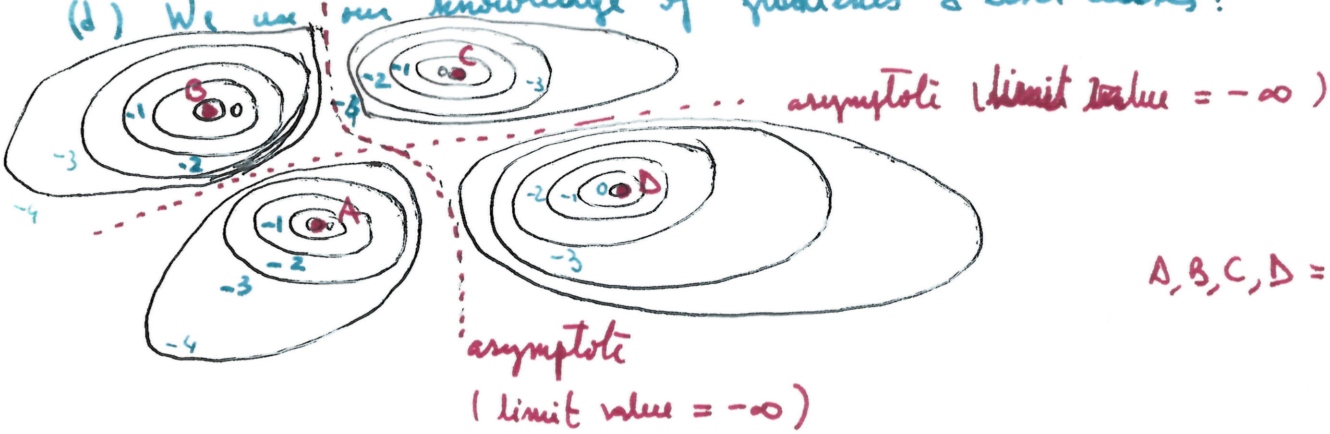
$f(x,y) = |y|$



$f_x \equiv 0$

but $\begin{cases} f_y^+(0,0) = 1 \\ f_y^-(0,0) = -1 \end{cases}$

(d) We use our knowledge of gradients & level curves:



A, B, C, D = local max

Problem 2: We write the velocity of the water:

$$W_{(x,y)} = \langle u_{(x,y)}, v_{(x,y)} \rangle$$

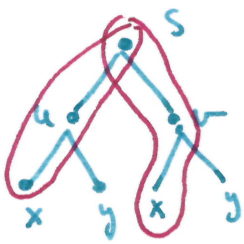
y = N-S direction

x = E-W direction

The speed of the water is the magnitude of $W(x,y)$, so $S_{(x,y)} = \sqrt{u^2_{(x,y)} + v^2_{(x,y)}}$

We write $S(u,v) = \sqrt{u^2+v^2}$ & note $\begin{cases} u = u_{(x,y)} = x(1+x)(-1+2y) \\ v = v(x,y) = y(y-1)(-1+2x) \end{cases}$

The rates of change of S are $\dfrac{\partial S}{\partial x}$ & $\dfrac{\partial S}{\partial y}$. We compute them using the Chain Rule:



contrib of each branch with leaf x

$$\begin{cases} \dfrac{\partial S}{\partial x} = S_u \cdot u_x + S_v \, v_x \\[2ex] \dfrac{\partial S}{\partial y} = S_u \, u_y + S_v \, v_y \end{cases}$$

$$\begin{cases} S_u = \dfrac{1}{2\sqrt{u^2+v^2}} \cdot 2u = \dfrac{u}{\sqrt{u^2+v^2}} \\[2ex] S_v = \dfrac{v}{\sqrt{u^2+v^2}} \end{cases} \qquad \begin{cases} u_x = (1+2x)(-1+2y) \\ u_y = 2x(1+x) \end{cases} \qquad \begin{cases} v_x = 2y(y-1) \\ v_y = (2y-1)(-1+2x) \end{cases}$$

$$S_{(x,y)} = (4x^4y^2 + 4x^2y^4 - 4x^4y + 8x^3y^2 - 8x^2y^3 - 4xy^2 + x^4 - 8x^3y + 8y^2x^2 + 8xy^3 + y^4 + 2x^3 - 4x^2y - 4xy^2 - 2y^3 + x^2 + y^2)^{1/2}$$

So $$\frac{\partial S}{\partial x} = \frac{1}{\sqrt{x^2(1+x)^2(-1+2y)^2 + y^2(y-1)^2(-1+2x)^2}} \left(((1+2x)(x(1+x))(-1+2y)^2 + (y^2(y-1)^2(-1+2x) \cdot 2 \right)$$

$$= \frac{1}{S_{(x,y)}} \left(4x^4 + 8x^3y^2 - 8x^3y + 12x^2y^2 - 8xy^3 - 12x^2y + 8xy^2 + 2x^3 + 4y^3 + 3x^2 - 2y^4 + x - 4yx - 2y^2 \right)$$

$$\frac{\partial S}{\partial y} = \frac{1}{S_{(x,y)}} \left((x(1+x))^2(-1+2y)(2) + (-1+2x)^2 \, y(y-1)(2y-1) \right)$$

$$= \frac{1}{S_{(x,y)}} \left(4x^4y + 8x^2y^3 - 2x^4 + 8x^3y - 12x^2y^2 - 8xy^3 - 4x^3 + 8x^2y + 12xy^2 + 2y^3 - 2x^2 - 4xy - 3y^2 + y \right)$$

$$= \frac{1}{S_{(x,y)}} (2y-1)\left(2x^4 + 2x^2y^2 + 4x^3 - 4x^2y - 4xy^2 + 2x^2 + 4xy + y^2 - y \right)$$

Problem 3: (a) $D_{\vec{u}}(f) = \nabla f \cdot \vec{u}$

$$\left.\begin{array}{l}\nabla f = \langle f_x, f_y\rangle = \langle 9x^2y, 3x^3-1\rangle \\ \vec{u} = \frac{1}{\sqrt{5}}(-2,-1)\end{array}\right\} \Rightarrow D_{\vec{u}}(f) = \boxed{\frac{-1}{\sqrt{5}}(18x^2y+3x^2-1)}$$

(b) By a Theorem discussed in class, the direction of maximum increase for f at (1,1,1) is $\nabla f_{(1,1,1)} = \langle f_{x(1,1,1)}, f_{y(1,1,1)}, f_{z(1,1,1)}\rangle$

$\nabla f = \langle yz^2, xz^2, 2xyz\rangle \rightsquigarrow \nabla f_{(1,1,1)} = \langle 1,1,2\rangle$

Since $\vec{u}$ must be a unit vector, then $\vec{u} = \frac{\langle 1,1,2\rangle}{|\langle 1,1,2\rangle|} = \boxed{\frac{\langle 1,1,2\rangle}{\sqrt{6}}}$

(c) The normal direction is $\langle -f_{x(e,1)}, -f_{y(e,1)}, 1\rangle$

Pt $= \langle e, 1, 1\rangle$ lies in the graph of f. $f(e,1) = \frac{\ln(e)}{e^1} = \frac{1}{e}$

(d) We use implicit differentiation: We assume $z = z(x,y)$ locally near $(1,0)$.

Take $\frac{\partial}{\partial x}$ of the implicit equation & notice x & y are independent

We get $\quad y + y\,2z\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0 \quad$ (by the product rule)

$$\frac{\partial z}{\partial x}(2yz+x) + z + y = 0$$

$$(*)\quad \boxed{\frac{\partial z}{\partial x} = \frac{-z-y}{x+2yz}}$$

at the point (1,0). we know $z = z(1,0)$ can be recovered from substituting $x=1$ & $y=0$ in the implicit equation

$1\cdot 0 + 0\cdot z^2 + 1\cdot z = 7 \quad$ so $\boxed{z=7}$

Notice that $(x+2yz)_{(1,0,7)} = 1 \neq 0$ so the expression $\frac{\partial z}{\partial x}$ in the

(*) above is well-defined

$$\frac{\partial z}{\partial x}(1,0) = \frac{-7-0}{1} = \boxed{-7}.$$

(e) We use linear approximation: $z = f(x,y) = xy^2 - x^2 + y$.

$$f(1.1, 1.9) = f(1,2) + f_x(1,2)(1.1-1) + f_y(1,2)(1.9-2)$$

$$f_x = y^2 - 2x \Rightarrow f_x(1,2) = 4 - 2 = 2$$

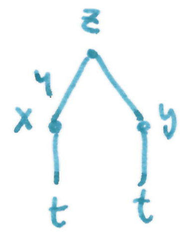
$$f_y = 2xy + 1 \Rightarrow f_y(1,2) = 4 + 1 = 5.$$

$$f(1,2) = 4 - 1 + 2 = 5$$

so $f(1.1, 1.9) = 5 + 2(0.1) + 5(-0.1) = 5 - 3(0.1) = \boxed{4.7}$.

The change is $\boxed{-0.3}$.

(f) We use the chain Rule



$$z_t = z_x \cdot x_t + z_y \cdot y_t$$

$z = x^2 - 3y^2 + 20 \Rightarrow z_x = 2x \Rightarrow \begin{cases} z_x(\cos t, \sin t) = 2\cos t \\ \Rightarrow z_x(\cos \pi/4, \sin \pi/4) = 2\cos \pi/4 = 2\frac{\sqrt{2}}{2} \end{cases}$

$z_y = -6y \Rightarrow \begin{cases} z_y(\cos t, \sin t) = -6\sin t. \\ \Rightarrow z_y(\cos \frac{\pi}{4}, \sin \frac{\pi}{4}) = -6\sin\frac{\pi}{4} = \frac{-6\sqrt{2}}{2} \end{cases}$

$$x_t = -\sin t \Rightarrow x_t\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$y_t = \cos t \Rightarrow y_t\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Conclusion: $\frac{\partial z}{\partial t}(\pi/4) = (\sqrt{2})\left(\frac{-\sqrt{2}}{2}\right) + \left(\frac{-6\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{-2}{2} - 3\cdot\frac{2}{2} = \boxed{-4}$

Problem 4: We start by finding the critical points inside the ellipse.

STEP 1: f is differentiable up to any order so the critical points are calculated as $\nabla f(x,y) = \vec{0}$.

$$\nabla f = \langle f_x, f_y \rangle = \langle 1 - y, -x \rangle$$

so $1 - y = 0$ & $-x = 0$ so $\boxed{x = 0}$

$\boxed{y = +1}$

So Only crit pt $= (0,1)$

$y^2 + 9x^2 = 1 + 0 = 1 < 9$
so it lies in the region!

We use the 2^{nd} Derivative Test to decide the nature of the critical point.

$f_{xx} = (1-y)_x = 0$

$f_{yy} = (1-x)_y = 0$ $\Rightarrow D_{(0,1)} = 0\cdot 0 - (-1)(-1) = -1 < 0$

$f_{xy} = (1-y)_y = -1$ $\Rightarrow (0,1)$ is a saddle point!

$f_{yx} = (1-x)_x = -1$

There are no other saddle points nor local max/min values in the interior of the ellipse.

STEP 2: We compute max/min values of f subject to the constraint

$g(x,y) = 9x^2 + y^2 = 9$ (boundary of the ellipse)

We find these max/min values using Lagrange multipliers

We must find (x, y, λ) that solve 3 equations:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 9 \end{cases} \iff \begin{cases} 1-y = \lambda(18x) & (1) \\ -x = \lambda(2y) & (2) \\ 9x^2 + y^2 = 9 & (3) \end{cases}$$

From (2) we obtain: $\lambda = \frac{-x}{2y}$ if $y \neq 0$

Substitute in (1) : $1 - y = \frac{-x}{2y}(18x)$ so $2y(1-y) = -18x^2$

$-y(1-y) = 9x^2$

We replace in (3): $9x^2 + y^2 = -y(1-y) + y^2 = 2y^2 - y = 9$

So $2y^2 - y - 9 = 0 \rightarrow y = \frac{1 \pm \sqrt{1 + 4\cdot 2\cdot 9}}{2\cdot 2} = \frac{1 \pm \sqrt{73}}{4}$

$x^2 = \frac{9 - y^2}{9} = 1 - \frac{1}{9\cdot 16}(1 + 73 \pm 2\sqrt{73}) = \frac{9\cdot 16 - (74 \pm 2\sqrt{73})}{9\cdot 16} = \frac{68 \mp 2\sqrt{73}}{9\cdot 16}$

$x^2 = \frac{34 \mp \sqrt{73}}{72} > 0$ so only + sign gives a valid solution

$x = \pm\sqrt{\frac{34+\sqrt{73}}{72}}$ & $y = \frac{1-\sqrt{73}}{4}$

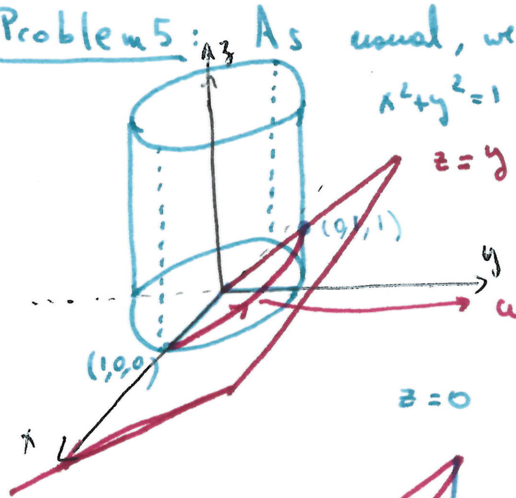
STEP 3: Compare values of f at these 2 points $= (a,b), (-a,b)$ $\quad a = \sqrt{\frac{34+\sqrt{73}}{72}}$, $b = \frac{1-\sqrt{73}}{4}$

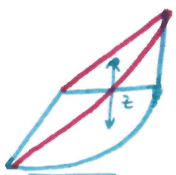
$f(a,b) = a(1-b) = \underset{>0}{a}\underset{>0}{\left(\frac{3+\sqrt{73}}{4}\right)} > 0 \Rightarrow$ MAX value.

$f(-a,b) = -a\left(\frac{3+\sqrt{73}}{4}\right) < 0 \longrightarrow$ MIN value.

• $(0,1)$ was saddle so no need to compare it!

Problem 5: As usual, we start by drawing the picture of the solid.



We get  $0 \leq z \leq y$.

Projection is $R=$  $0\leq x^2+y^2\leq 1 \rightsquigarrow \begin{cases} 0\leq y\leq \sqrt{1-x^2} \\ 0\leq x\leq 1\end{cases}$ $0\leq x\leq 1$

$$Vol(D) = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y 1\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} (y-0)\,dy\,dx = \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

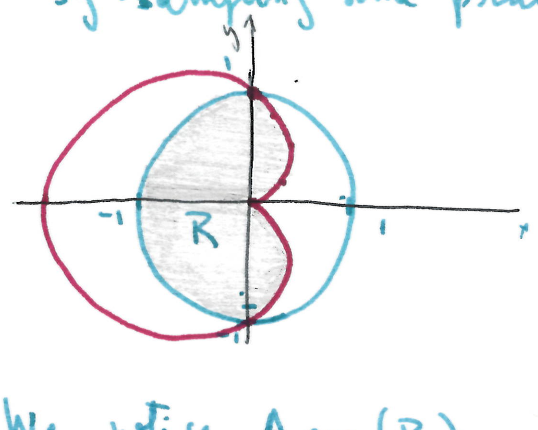
$$= \int_0^1 \frac{(1-x^2)}{2}\,dx = \frac{x}{2}-\frac{x^3}{6}\Big|_{x=0}^{x=1} = \frac{1}{2}-\frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

- In polar coordinates: $R = \begin{matrix} 0\leq r\leq 1 \\ 0\leq\theta\leq\frac{\pi}{2}\end{matrix}$ $\quad 0\leq z\leq y = r\sin\theta$.

$$Vol = \int_0^{\pi/2}\int_0^1\int_0^{r\sin\theta} 1\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/2} \frac{r^3}{3}\sin\theta\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\sin\theta\,d\theta = \frac{-\cos\theta}{3}\Big|_{\theta=0}^{\theta=\pi/2} = \frac{-(0-1)}{3} = \boxed{\frac{1}{3}}$$

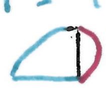
Problem 6: (a) We start by drawing the region. We draw the cardioid by sampling some points (θ in $[0,2\pi]$. Notice that the values at θ & $2\pi-\theta$ will be the same)



For θ in $[0,\frac{\pi}{2}]$ & θ in $[\frac{3\pi}{2}, 2\pi]$ we have $0\leq\cos\theta\leq 1$ so $0\leq r = 1-\cos\theta\leq 1$

But for θ in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ $\quad r = 1-\cos\theta \geq 1$.

and it's $r=1$ only for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

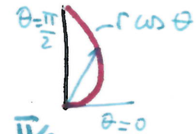
We notice Area$(R) = 2$ Area() $= 2$(Area() + Area())

$\text{Area}(◺) = \text{Area}(\tfrac{1}{4}\text{ circle}) = \tfrac{1}{4}\text{Area}(\text{unit circle}) = \tfrac{1}{4}(\pi\, 1^2) = \tfrac{\pi}{4}.$

Area(D) can be computed in polar coordinates.

Region is defined as $0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1-\cos\theta$



$$\text{Area}(D) = \int_0^{\pi/2}\int_0^{1-\cos\theta} 1\cdot r\, dr\, d\theta = \int_0^{\pi/2} \frac{r^2}{2}\Big|_{r=0}^{r=1-\cos\theta} d\theta = \int_0^{\pi/2} \frac{(1-\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1+\cos^2\theta - 2\cos\theta}{2}\, d\theta = \int_0^{\pi/2} \frac{1}{2} + \frac{1}{4} + \frac{\cos 2\theta}{4} - \cos\theta\, d\theta$$

use $\cos^2\theta - \sin^2\theta = \cos(2\theta)$

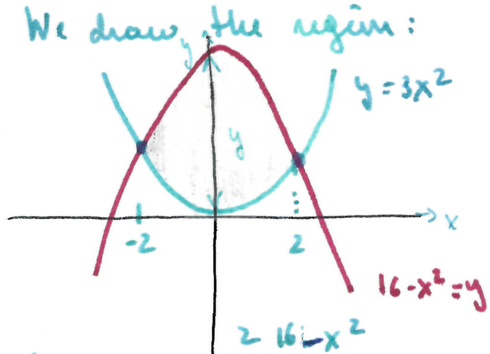
$\cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$ $\Big\}\ \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int_0^{\pi/2} \frac{3}{4} - \cos\theta + \frac{\cos 2\theta}{2}\, d\theta = \frac{3\theta}{4} - \sin\theta + \frac{\sin 2\theta}{4}\Big|_0^{\theta=\pi/2}$$

$$= \frac{3}{4}\frac{\pi}{2} - 1 + 0 - \left(0 - 0 + \frac{0}{4}\right) = \frac{3\pi}{8} - 1$$

Conclusion $\text{Area}(R) = 2\left(\frac{\pi}{4} + \frac{3\pi}{8} - 1\right) = 2\left(\frac{5\pi}{8} - 1\right) = \boxed{\frac{5\pi}{4} - 2}$

(b) We draw the region:



Find the intersections

$3x^2 = 16 - x^2$

$4x^2 = 16$

$x^2 = 4$ so $x = \pm 2$ & $y = 16 - 4 = 12$.

It's a type I region.

$$\text{Area} = \int_{-2}^{2}\int_{3x^2}^{16-x^2} 1\, dy\, dx = \int_{-2}^{2} -3x^2 + (16-x^2)\, dx = \int_{-2}^{2} -4x^2 + 16\, dx = -\frac{4}{3}x^3 + 16x\Big|_{x=-2}^{x=2}$$

$$= \left(-\frac{4}{3}8 + 16\cdot 2\right) - \left(-\frac{4}{3}(-8) + 16(-2)\right) = 2\cdot\left(-\frac{4}{3}8 + 16\cdot 2\right) = 2\left(-\frac{32}{3} + 32\right)$$

$$= \frac{2\cdot 32\cdot 2}{3} = \boxed{\frac{128}{3}}$$