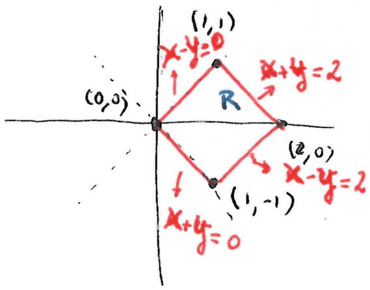
Math 2153 – April 14, 2016 Full Name: ____________________

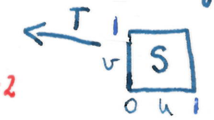
Quiz 4

1. *(1 point)* Evaluate the double integral $\iint_R xy\,dA$, where R is the square with vertices $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ using the change of variables $x = u+v$, $y = u-v$.



$2u = x+y$
$2v = x-y$

so $T: \begin{cases} x = u+v \\ y = u-v \end{cases}$ $\qquad J_{(u,v)} = \begin{vmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \\ 1 & 1 & x \\ 1 & -1 & y \end{vmatrix} = -1-1 = -2$



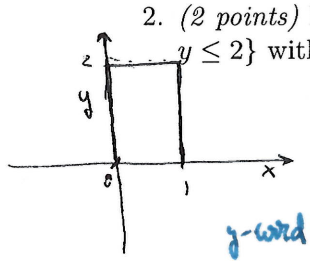
$S.\begin{cases} 0 \le 2u \le 2 \\ 0 \le 2v \le 2 \end{cases}$

Using this change of variables we get

$\iint_R xy\,dA = \iint_S (u+v)(u-v)\,|\underset{2}{-2}|\,dA = 2\int_0^1\int_0^1 (u^2-v^2)\,du\,dv$

$= 2\int_0^1 u^2v - \frac{v^3}{3}\Big|_{v=0}^{v=1} du = 2\int_0^1 u^2 - \frac{1}{3}\,du = 2\left(\frac{u^3}{3} - \frac{u}{3}\Big|_0^1\right) = \boxed{0}$

2. *(2 points)* Find the y-coordinate of the centroid of the region $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 2\}$ with density function $\rho(x,y) = 1 + x/2$.



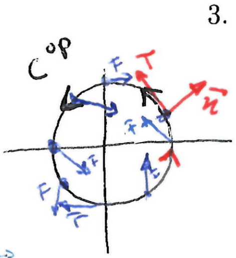
Mass $= \iint_R \rho(x,y)\,dA = \int_0^1\int_0^2 \left(1+\frac{x}{2}\right) dy\,dx = \int_0^1 \left(1+\frac{x}{2}\right)2\,dx = 2\left(x+\frac{x^2}{4}\right)\Big|_0^1 = \boxed{\frac{5}{2}}$

y-coord centroid $= \frac{1}{5/2}\iint_R y\,\rho(x,y)\,dA = \frac{1}{5/2}\int_0^1\int_0^2 y\left(1+\frac{x}{2}\right)dy\,dx = \frac{2}{5}\int_0^1 \frac{y^2}{2}\left(1+\frac{x}{2}\right)\Big|_{y=0}^{y=2} dx$

$= \frac{2}{5}\int_0^1 2\left(1+\frac{x}{2}\right)dx = \frac{2}{5}\cdot\frac{5}{2} = \boxed{1}$ → This ~~was~~ expected because the density function is indep of y & the rectangle is symmetric about the $y=1$ line.

Conclusion centroid has y-coord $= 1$.

3. *(2 points)* Find the circulation and flux of the vector field $\mathbf{F}(x,y) = \langle y-x, x\rangle$ on the circle of radius 2 centered at the origin, traversed **clockwise**.



$\vec{n} = \frac{\langle x, y\rangle}{2}_{(x,y)}$

Counter clockwise param: $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ $\quad 0 \le t \le 2\pi$ → parameterizes C^{op}.
$\vec{r}\,' = 2\langle -\sin t, \cos t\rangle$

Recall: $\oint_C \mathbf{F}\cdot\vec{T}\,ds = -\oint_{C^{op}} \mathbf{F}\cdot\vec{T}\,ds = -\int_0^{2\pi} 2\langle 2\sin t - 2\cos t, 2\cos t\rangle\cdot\langle -\sin t, \cos t\rangle dt$

$= -2\int_0^{2\pi} -2\sin^2 t + 2\cos t\sin t + 2\cos^2 t\,dt = -2\int_0^{2\pi} 2\underbrace{(\cos^2 t - \sin^2 t)}_{=\cos 2t} + 2\cos t\sin t\,dt$

$= -2\left(\sin 2t + \sin^2 t\right)\Big|_{t=0}^{t=2\pi} = \boxed{0}$ ⇒ $\boxed{\text{circulation is } 0}$ ← consistent w/ our calculation

$\cos^2 t + \sin^2 t = 1$
$\cos^2 t - \sin^2 t = \cos 2t$
$2\cos^2 t = 1 + \cos 2t$

Notice: F is conservative ($f_y = 1 = g_x$ & disk is open & simply conn) so $\oint_C \mathbf{F}\cdot d\vec{r} = 0$.

Flux: $\oint_{C^{op}} \mathbf{F}\cdot\vec{n}\,ds = \int_0^{2\pi} \langle 2\sin t - 2\cos t, 2\cos t\rangle\cdot\langle 2\cos t, 2\sin t\rangle dt = \int_0^{2\pi} 4\cos t\sin t - 4\cos^2 t + 4\sin t\cos t\,dt$

$= \int_0^{2\pi} 4(2\cos t\sin t) - 2(1+\cos 2t)\,dt = \left(4\sin^2 t - 2t - \sin 2t\right)\Big|_{t=0}^{t=2\pi} = 0 - 4\pi \to 4\pi$ ⇒ Flux$(F, C) = \boxed{+4\pi}$