Recitation III (1/28/16)

Problem 1:

(a) From P & Q we get the direction $\vec{v} = \overrightarrow{PQ} = \langle 2,-2,-3 \rangle$

Param eqns : $\begin{cases} x = 1 + t\,2 \\ y = 2 + t(-2) \\ z = 2 + t(-3) \end{cases}$

(b) $\vec{r}(t) = \overrightarrow{OP} + t\,\vec{v}$ and $\vec{r}$ is parallel to $\langle 2,-3,-4 \rangle$, so we can take $\vec{r} = \langle 2,-3,-4 \rangle$. Also, $P = (-1,1,3)$.

So $\vec{r}(t) = \langle -1, 1, 3 \rangle + t \langle 2,-3,-4 \rangle$

Problem 2: (a) We must find values for t & s such that

head of $(\vec{r}(t))$ = intersection pt = head of $(\vec{q}(s))$

So $\vec{r}(t) = \langle 1+t, 2-t, 3-3t \rangle = \langle -1+s, 2-2s, -2-s \rangle = q(s)$

Equivalently : $\begin{cases} 1+t = -1+s & \Leftrightarrow & 2+t = s \\ 2-t = 2-2s & \Leftrightarrow & 2s = t \\ 3-3t = -2-s & \Leftrightarrow & 5 = 3t - s \end{cases}$ $\rightarrow$ CHECK : $4 \neq 1$.

$\Rightarrow 5 = 6s - s = 5s$ so $\boxed{s=1}$ & $\boxed{t=2}$

Conclusion: The lines do not intersect (because no t,s satisfy all 3 equations!)

Note: $\vec{r}$ and $\vec{q}$ are not parallel since their directions are not parallel (direction of $\vec{r} = \langle 1,-1,-3 \rangle$)
direction of $q = \langle 1,-2,-1 \rangle$

Therefore, $\vec{r}$ are not parallel & don't meet, so they are skew lines.

(b) Same procedure : We must find t & s satisfying all 3 equations
$\begin{cases} 3+t = 4+s & (*) \\ 2-t = 2-2s \\ 3-3t = -2-s \end{cases}$ $\rightarrow s=1$ & $t=2$

Next, we check $(*)$: $3+2 = 4+1$ ✓

So there is a unique intersection pt = head of $(\vec{r}(2)) = (5,0,-3)$.

Problem 3:

Recall: How to draw curves in space? We have some Tools at hand:

TOOL 0: Plot some points

TOOL 1: Use projections to the 3 coordinate planes, draw the curve in 2 space and try to "lift" the result.

TOOL 2: Find (polynomial) relations among the coordinates of $\vec{r}$ to realize the curve as an intersection of 2 surfaces in 3-space

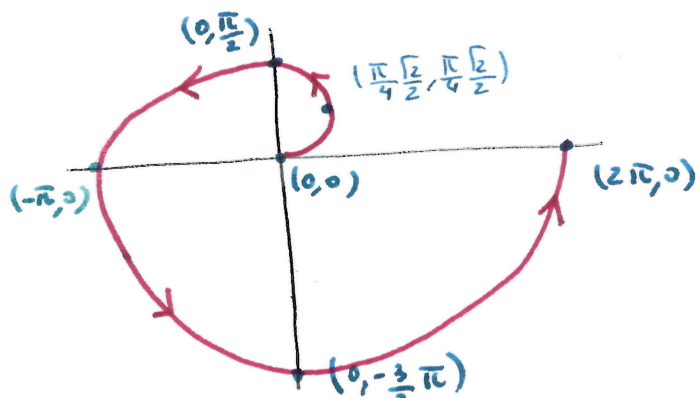
TOOL 3: Detect possible oscillations (eg helix curve [page 835], & many homework problem)

In our case: $\vec{r}(t) = \langle \underline{t\cos t}, \underline{t\sin t}, t \rangle$ for $t \geq 0$.

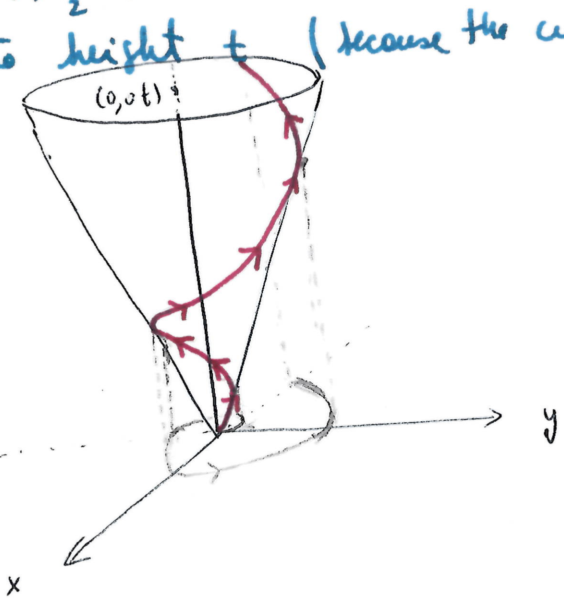
T2: Equation: $x^2 + y^2 = z^2$ This describes an icecream cone

The curve lives in this cone!

T1: Project to the xy-plane: we get a spiral! We plot some points $t = 0, \frac{\pi}{4}, \pi/2, \pi, \frac{3\pi}{2}, 2\pi$



We lift each point to height t (because the curve lies in the cone)



Problem 4: • If the $\lim_{t\to 1} f(t)$ does not exist, then the limit doesn't exist for $\vec{r}(t)$ either.

• If the limit exists by $\lim_{t\to 1} f(t) \neq f(1)$, then either $\lim_{t\to 1} \vec{r}(t)$ does not exist (if one of the limits for $g(t)$ or $h(t)$ doesn't), or if it does, then $\lim_{t\to 1} \vec{r}(t) = \langle \lim_{t\to 1} f(t), \lim_{t\to 1} g(t), \lim_{t\to 1} h(t) \rangle$ cannot be $\vec{r}(1)$ because its first component is not $f(1)$.

Conclusion: $\vec{r}$ cannot be continuous at $t=1$.

Problem 5: By definition $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ if $\vec{r}\,'(t) \neq 0$.

We compute $\vec{r}\,'(t) = \langle \frac{1}{t}, \frac{-1}{t^2}, \frac{-2}{t^3} \rangle$ valid on $t>0$ ✓

$$|\vec{r}\,'(t)| = \sqrt{\frac{1}{t^2} + \frac{1}{t^4} + \frac{4}{t^6}} = \frac{1}{t^3}\sqrt{t^4+t^2+4}$$

Then: $\vec{r}\,'(t) \neq 0$ for every $t>0$,

so $\vec{T}(t)$ is defined for every $t>0$ and

$$\vec{T}(t) = \frac{\langle \frac{1}{t}, -\frac{1}{t^2}, \frac{-2}{t^3} \rangle}{\frac{\sqrt{t^4+t^2+4}}{t^3}} = \frac{\langle t^2, -t, \frac{-2}{t} \rangle}{\sqrt{t^4+t^2+4}}$$

The domain is $\{t>0\}$ and $\vec{T}(t)$ is continuous on this domain.

Problem 6: We solve by computing antiderivatives and using the given initial conditions: $\vec{r}(0) = \langle 2, 0 \rangle$, $\vec{v}(0) = \langle 3, -3 \rangle$

$\vec{a}(t) = \langle t^2, e^t \rangle$ $t \geq 0$.

(a) $\vec{v}(t) = \int a(s)\, ds = \langle \frac{t^3}{3}, e^t \rangle + \vec{C}$ $\qquad \vec{C}_1 = \langle c_1, c_2 \rangle$

$\langle 3, -3 \rangle = \vec{v}(0) \overset{?}{=} \langle 0, e^0 \rangle + \vec{C} = \langle c_1, 1+c_2 \rangle \implies c_1 = 3,\ c_2 = -4$

So $\vec{r}(t) = \langle 3 + \frac{t^3}{3}, -4 + e^t \rangle$.

(b) $\vec{r}(t) = \int \vec{v}(s)\, ds = \langle 3t + \frac{t^4}{12}, -4t + e^t \rangle + \vec{C}_2$, so

$\langle 2, 0 \rangle = \vec{r}(0) = \langle 0, 1 \rangle + \vec{C}_2 \implies \vec{C}_2 = \langle 2, -1 \rangle$ & $\boxed{\vec{r}(t) = \langle 2+3t+t^4/12, -4t+e^t-1 \rangle}$

Problem 7: Only force acting is gravity, so $\vec{a} = <0, -g>$
$= <0, -9.8>$

Initial conditions: $\vec{r}_{(0)} = <0, 1>$

$\vec{v}_{(0)} = ?$ angle $= 0$ & speed $= 2$ m/s
so $\vec{v}_{(0)} = \langle 2, 0 \rangle$.

- We integrate to find $\vec{v}_{(t)}$ & $\vec{r}_{(t)}$

$\vec{v}_{(t)} = \int \vec{a}_{(s)}\, ds = <0, -9.8t> + \vec{C}$, so $\vec{C} = <2, 0>$

Then $\vec{v}_{(t)} = <2, -9.8t>$

$\vec{r}_{(t)} = \int \vec{v}_{(s)}\, ds = \langle 2t, -\frac{9.8t^2}{2} \rangle + \vec{C_2}$ so $\vec{C_2} = <0, 1>$

Then $\vec{r}(t) = <2t, -4.9t^2 + 1>$

- The distance d is achieved when the y-component of $\vec{r}_{(t_0)} = 0$.

so $t_0^2 = \frac{1}{4.9} \Rightarrow t_0 = \sqrt{\frac{1}{4.9}} = \frac{\sqrt{10}}{7}$

distance d = x-component of $\vec{r}(t_0)$

$= 2\frac{\sqrt{10}}{7}$.