Recitation IV (2/4/16)

Problem 1: (a) We solve by integrating:

$$\vec{r}(t) = \int \langle \sqrt{2}t, 1, -t^2 \rangle \, dt = \langle \tfrac{\sqrt{2}}{2}t^2, t, \tfrac{-t^3}{3} \rangle + \vec{C}$$

Use $\langle 1,1,1 \rangle = \vec{r}(0) = \vec{0} + \vec{C}$, to conclude

$$\boxed{\vec{r}(t) = \langle 1+\tfrac{\sqrt{2}}{2}t^2, 1+t, -\tfrac{t^3}{3}+1 \rangle}.$$

(b) By definition:
$$L = \int_0^6 |\vec{r}\,'(t)| \, dt = \int_0^6 \sqrt{(\sqrt{2}t)^2 + 1^2 + (-t^2)^2} \, dt$$
$$= \int_0^6 \sqrt{t^4 + 1 + 2t^2} \, dt = \int_0^6 \sqrt{(1+t^2)^2} \, dt = \int_0^6 (1+t^2) \, dt$$
$$= \left(t + \frac{t^3}{3}\right)\Bigg|_{t=0}^{t=6} = (6+72) - 0 = \boxed{78}.$$

(c) We can compute the curvature in several ways:

OPTION 1: $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$

- $|\vec{r}\,'(t)| = 1+t^2$ (done in b))
- $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle \sqrt{2}t, 1, -t^2 \rangle}{1+t^2} \Rightarrow \dfrac{d\vec{T}}{dt} = \left\langle \dfrac{\sqrt{2}(1+t^2) - \sqrt{2}t\,2t}{(1+t^2)^2}, \dfrac{-2t}{(1+t^2)^2}, \dfrac{-2t(1+t^2) + t^2 2t}{(1+t^2)^2} \right\rangle$

$$\Rightarrow \frac{d\vec{T}}{dt} = \frac{1}{(1+t^2)^2} \langle \sqrt{2} - \sqrt{2}t^2, -2t, -2t \rangle$$

So
$$|\vec{T}'(t)| = \frac{1}{(1+t^2)^2} \sqrt{(\sqrt{2}(1-t^2))^2 + 4t^2 + 4t^2} = \frac{\sqrt{2(1+t^4-2t^2)+8t^2}}{(1+t^2)^2}$$
$$= \frac{\sqrt{2}\sqrt{1+t^4+2t^2}}{(1+t^2)^2} = \frac{\sqrt{2}(1+t^2)}{(1+t^2)^2} = \frac{\sqrt{2}}{1+t^2}.$$

Then $K(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{\sqrt{2}/1+t^2}{1+t^2} = \dfrac{\sqrt{2}}{(1+t^2)^2}$

<u>OPTION 2</u>: $K(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$.

$\vec{r}''(t) = \langle \sqrt{2}, 0, -2t \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2}t & 1 & -t^2 \\ \sqrt{2} & 0 & -2t \end{vmatrix} = (-2t)\vec{i} - (-2\sqrt{2}t^2 + \sqrt{2}t^2)\vec{j} + (-\sqrt{2})\vec{k}$$

$= \langle -2t, \sqrt{2}t^2, -\sqrt{2} \rangle$

Then $|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{4t^2 + 2t^4 + 2} = \sqrt{2}\sqrt{t^4 + 1 + 2t^2} = \sqrt{2}(1+t^2)$

So, $K(t) = \dfrac{\sqrt{2}(1+t^2)}{(1+t^2)^3} = \boxed{\dfrac{\sqrt{2}}{(1+t^2)^2}}$ (agrees with OPTION 1 ✓).

• The value of $K(t)$ is <u>largest when $t=0$</u>, so $K(0) = \sqrt{2}$.

d) In order to compute the torsion, we need to know the TNB-frame

• $\vec{N}(t) = \dfrac{T'(t)}{|\vec{T}'(t)|} = \dfrac{\frac{1}{(1+t^2)^2}\langle \sqrt{2}(1-t^2), -2t, -2t \rangle}{\sqrt{2}/(1+t^2)}$

$= \dfrac{1}{(1+t^2)} \langle 1-t^2, -\sqrt{2}t, -\sqrt{2}t \rangle$

• $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \dfrac{1}{(1+t^2)^2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2}t & 1 & -t^2 \\ (1-t^2) & -\sqrt{2}t & -\sqrt{2}t \end{vmatrix}$

$= \dfrac{1}{(1+t^2)^2} \left((-\sqrt{2}t - \sqrt{2}t^3)\vec{i} - (-2 \cdot t^2 + t^2(1-t^2))\vec{j} + (-2t^2 - (1-t^2))\vec{k} \right)$

$= \dfrac{1}{(1+t^2)^2} \langle -t\sqrt{2}(1+t^2), +t^2(1+t^2), -(1+t^2) \rangle = \dfrac{\langle -\sqrt{2}t, t^2, -1 \rangle}{1+t^2}$.

$$\frac{d\vec{B}}{dt} = \frac{\langle -\sqrt{2}(1+t^2)+\sqrt{2}t(2t),\ 2t(1+t^2)-t^2 2t,\ 2t \rangle}{(1+t^2)^2} =$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}}{dt}\Big/(1+t^2) = \frac{\langle \sqrt{2}(t^2-1),\ 2t,\ 2t \rangle}{(1+t^2)^3}$$

$$\tau_{(t)} = -\frac{d\vec{B}}{ds}\cdot \vec{N}_{(t)} = \frac{-1}{(1+t^2)^4}\left\{(1-t^2)\sqrt{2}(t^2-1) + 2t(-\sqrt{2}t) + 2t(-\sqrt{2}t)\right\}$$

$$= \frac{-\sqrt{2}}{(1+t^2)^4}\left(-(t^4-2t^2+1) - 4t^2\right) = \frac{+\sqrt{2}}{(1+t^2)^4}(t^4+2t^2+1)$$

$$= \frac{\sqrt{2}}{(1+t^2)^4}(1+t^2)^2 = \boxed{\frac{\sqrt{2}}{(1+t^2)^2}}$$

The value of the torsion is largest when $t=0$, so $\tau_{(0)} = \sqrt{2}$.

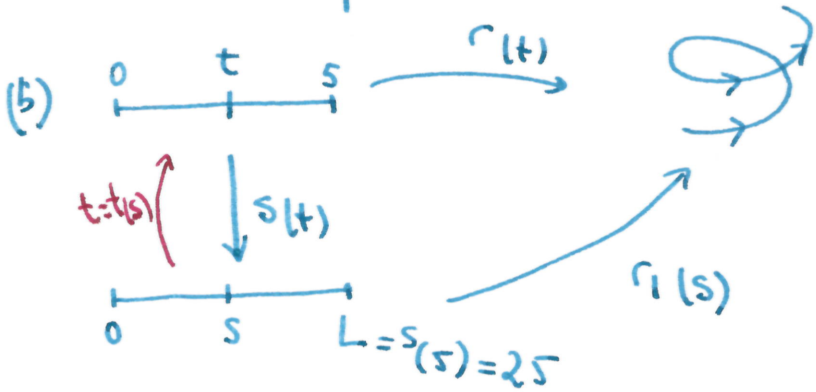
Problem 2: (a) To answer this question, we compute the length function

$$s(t) = \int_0^t |\vec{r}\,'(q)|\,dq = \int_0^t 5\,dq = 5q\,\Big|_{q=0}^{q=t} = 5t.$$

$$\vec{r}\,'(q) = \langle 3\cos q,\ 4,\ -3\sin q \rangle \Rightarrow |\vec{r}\,'(q)| = \sqrt{9+16} = 5$$

We want to find t such that $s(t) = 5$, so $\boxed{t=1}$.

(b)



We want to find $r_1(s)$ so we write $t = t(s)$ as a function of s and use $r_1(s) = r(t(s)) = \langle 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}, 3\cos\left(\frac{s}{5}\right) \rangle$

$s = 5t \Rightarrow t = \frac{s}{5} = t(s)$ is the arc length parameterization.

Problem 3 $\vec{r}(t) = \langle \ln t, \frac{1}{t}, t \rangle$ so $\vec{r}'(t) = \langle \frac{1}{t}, \frac{-1}{t^2}, 1 \rangle$

$$\frac{ds}{dt}(t) = |\vec{r}'(t)| = \sqrt{\frac{1}{t^2} + \frac{1}{t^4} + 1} = \sqrt{\frac{t^2+1+t^4}{t^2}} = \frac{\sqrt{1+t^2+t^4}}{t}$$

$$\left.\frac{ds}{dt}\right|_{t=2} = \frac{\sqrt{1+4+16}}{2} = \frac{\sqrt{21}}{2}.$$

Problem 4: (a) We compute the curvature with the formula $\frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$.

$\vec{r}'(t) = \langle -\sin t, \cos t, 2 \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{2+1} = \sqrt{3}$

$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (2\sin t)\vec{i} - (2\cos t)\vec{j} + (\sin^2 t + \cos^2 t)\vec{k}$$
$$= \langle 2\sin t, -2\cos t, 1 \rangle$$

$\Rightarrow |\vec{r}'(t) \times r''(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$

Conclusion: $\kappa(t) = \frac{\sqrt{5}}{(\sqrt{3})^3}$ is constant.

In order to compute the torsion we start by computing the TNB-frame.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{\sqrt{3}} \langle -\sin t, \cos t, 2 \rangle$$

$\vec{T}'(t) = \frac{1}{\sqrt{3}} \langle -\cos t, -\sin t, 0 \rangle$ & $|\vec{T}'(t)| = \frac{1}{\sqrt{3}}$.

So $\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{3}}(\vec{r}'(t) \times \vec{r}''(t)) = \frac{\langle 2\sin t, -2\cos t, 1 \rangle}{\sqrt{3}}$$

$$\frac{d\vec{B}}{dt} = \frac{\langle 2\cos t, -2\sin t, 0 \rangle}{\sqrt{3}} \Rightarrow \frac{d\vec{B}}{ds} = \frac{\langle 2\cos t, -2\sin t, 0 \rangle}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}$$
$$= \frac{1}{3} \langle 2\cos t, -2\sin t, 0 \rangle$$

$$\Rightarrow \tau(t) = -\frac{d\vec{B}}{ds} \cdot \vec{N}(t) = -\frac{1}{3}\left(2\cos^2 t + 2\sin^2 t + 0\right) = \boxed{\frac{-2}{3}}.$$

Conclusion: $\tau(t)$ is constant $= -2/3$.

(b) $\vec{r}(t) = \left(\operatorname{scal}_{\vec{T}(t)} \vec{r}(t)\right)\vec{T}(t) + \left(\operatorname{scal}_{\vec{N}(t)} \vec{r}(t)\right)\vec{N}(t) + \left(\operatorname{scal}_{\vec{B}(t)} \vec{r}(t)\right)\vec{B}$.

because $\vec{T}, \vec{N}, \vec{B}$ are mutually perpendicular & have magnitude 1.

- $\operatorname{scal}_{\vec{T}(t)} \vec{r}(t) = \vec{r}(t) \cdot \vec{T}(t) = \frac{1}{\sqrt{3}}(-\sin t\cos t + \cos t\sin t + 4t) = \frac{4t}{\sqrt{3}}$.
- $\operatorname{scal}_{\vec{N}(t)} \vec{r}(t) = \vec{r}(t) \cdot \vec{N}(t) = (-\cos^2 t - \sin^2 t + 0) = -1$.
- $\operatorname{scal}_{\vec{B}(t)} \vec{r}(t) = \vec{r}(t) \cdot \vec{B}(t) = \frac{1}{\sqrt{3}}(2\sin t\cos t - 2\cos t\sin t + 2t) = \frac{2t}{\sqrt{3}}$

$\vec{r}'(t) = \underbrace{|\vec{r}'(t)|}\,\vec{T}(t)$ so $\operatorname{scal}_{\vec{T}(t)} \vec{r}'(t) = |\vec{r}'(t)|$ & other two scalar components are 0.

$\vec{r}''(t) = \vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)$, so

$\operatorname{scal}_{\vec{B}(t)} \vec{a}(t) = 0$.

$\operatorname{scal}_{\vec{N}(t)} \vec{a}(t) = a_N(t) = K(t)\,|\vec{r}'(t)| = \boxed{\frac{\sqrt{5}}{3}}$

$\operatorname{scal}_{\vec{T}(t)} \vec{a}(t) = a_T(t) = \frac{d^2 s}{d^2 t} = \frac{d}{dt}(|\vec{r}'(t)|) = \frac{d}{dt}\sqrt{3} = 0$.

In this case, $\vec{r}''(t)$ is parallel to $\vec{N}(t)$.

<u>Problem 5</u>: By a theorem from Lecture X, we know that

$$\vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)$$

$$a_T(t) = \frac{d^2 s}{d^2 t} = \frac{d}{dt}(|\vec{r}'(t)|) = \frac{d}{dt}(|\langle 3t^2, 2t\rangle|) = \frac{d}{dt}\left(\sqrt{9t^4 + 4t^2}\right)$$

$$= \frac{1}{2}\,\frac{36t^3 + 8t}{\sqrt{9t^4 + 4t^2}} = \frac{16t^2 + 4}{\sqrt{4 + 9t^2}}.$$

$$a_N(t) = \kappa(t)\,|r'(t)|$$

$$r' = \langle 3t^2, 2t\rangle \Rightarrow |r'(t)| = \sqrt{9t^4+4t^2} = t\sqrt{9t^2+4}$$

Note that we cannot use the cross product formula to compute the curvature since the trajectory lies in $\mathbb{R}^2$.

$$\kappa(t) = \frac{\left|\frac{d}{dt}\vec{T}(t)\right|}{|\vec{r}\,'(t)|}$$

$$\vec{T}(t) = \frac{\langle 3t^2, 2t\rangle}{\sqrt{9t^4+4t^2}} = \frac{\langle 3t, 2\rangle}{\sqrt{9t^2+4}} \Rightarrow \frac{d\vec{T}(t)}{dt} = \frac{1}{\sqrt{9t^2+4}}\langle 3, 0\rangle + \frac{\frac{-18t}{2}}{\left(\sqrt{9t^2+4}\right)^3}\langle 3t, 2\rangle$$

$$\frac{d\vec{T}}{dt}(t) = \frac{\langle 3((9t^2+4) - 9t^2), -18t\rangle}{\left(\sqrt{9t^2+4}\right)^3} = \frac{\langle 12, -18t\rangle}{\left(\sqrt{9t^2+4}\right)^3}$$

$$\Rightarrow \left|\frac{d\vec{T}}{dt}(t)\right| = \frac{\sqrt{144 + 18^2t^2}}{\left(\sqrt{9t^2+4}\right)^3}$$

So $$\kappa(t) = \frac{\sqrt{144+18^2t^2}}{t\left(\sqrt{4+9t^2}\right)^4} = \frac{6\sqrt{4+9t^2}}{t\left(\sqrt{4+9t^2}\right)^4} = \frac{6}{t\left(\sqrt{4+9t^2}\right)^3}$$

and $$\boxed{a_N(t) = \frac{6}{t(4+9t^2)}}$$