Recitation V (2/11/16)

Problem 1: The osculating plane at time t is generated by the vectors $\vec{T}_{(t)}$ & $\vec{N}_{(t)}$ and passes through the point P = head of $\vec{r}(t)$. $(\vec{OP} = \vec{r}(t))$

The normal vector is $\vec{T}_{(t)} \times \vec{N}_{(t)} = \vec{B}_{(t)}$ by definition, so we only need $\vec{B}_{(1)}$ & $\vec{r}_{(1)} = \langle 1,1,1\rangle$ to find the equation: $\vec{PQ} \cdot \vec{B}_{(1)} = 0$.

- $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 3t^2, 2t, 1\rangle}{\sqrt{9t^4+4t^2+1}}$ $\qquad \dfrac{d\vec{T}}{dt} = \dfrac{\langle 6t, 2, 0\rangle}{\sqrt{9t^4+4t^2+1}} + \left[-\dfrac{1}{2}\dfrac{(36t^3+8t)}{\left(\sqrt{9t^4+4t^2+1}\right)^3}\langle 3t^2, 2t, 1\rangle \right.$

- $\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}(t)}{|\vec{T}\,'(t)|} = \dfrac{1}{|\vec{T}\,'(t)|}\left(\dfrac{\langle 6t,2,0\rangle}{\sqrt{9t^4+4t^2+1}} - \dfrac{\frac{1}{2}(36t^3+8t)\langle 3t^2,2t,1\rangle}{\left(\sqrt{9t^4+4t^2+1}\right)^3}\right)$

so $\vec{B}_{(1)} = \vec{T}_{(1)} \times \vec{N}_{(1)} = \dfrac{\langle 3,2,1\rangle}{\sqrt{14}} \times \dfrac{1}{|\vec{T}\,'_{(1)}|}\left(\dfrac{\langle 6,2,0\rangle}{\sqrt{14}} - \dfrac{1}{2}\dfrac{44\langle 3,2,1\rangle}{\sqrt{14}^3}\right)$

$= \dfrac{1}{|\vec{T}\,'_{(1)}|}\dfrac{1}{(\sqrt{14})^2}\left(\langle 3,2,1\rangle \times \langle 6,2,0\rangle\right)$ $\rightsquigarrow$ proportional to $\langle 3,2,1\rangle \times \langle 6,2,0\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 6 & 2 & 0 \end{vmatrix} = (-2)\,\vec{i} - (-6)\,\vec{j} + (-6+12)\,\vec{k}$$
$$= -2\vec{i} + 6\vec{j} - 6\vec{k} = 2\langle -1, 3, -3\rangle$$

We take $\vec{n} = \langle -1, 3, -3\rangle$ (scalar multiple of $\vec{B}_{(1)}$).

Equation: $\langle x-1, y-1, z-1\rangle \cdot \langle -1, 3, -3\rangle = 0$

$$-(x-1) + 3(y-1) - 3(z-1) = 0$$

$$\boxed{x - 3y + 3z = 1}$$

Problem 2: (a) To find the line of intersection, we combine the equations for both planes

$$\begin{cases} x + 2y - z = 3 & \Rightarrow \boxed{z = x + 2y - 3} \quad (1) \\ 2x + y + z = 9 & (2) \end{cases}$$

Replace value of z in (2): $2x + y + x + 2y - 3 = 9$

$$3x + 3y = 12$$
$$x + y = 4 \Rightarrow \boxed{y = 4 - x} \quad (*)$$

Replace (*) back in (1) $\quad z = x + 2(4-x) - 3$

$$z = -x + 5$$

So $\langle x, y, z\rangle = \langle x, 4-x, -x+5\rangle = x\langle 1,-1,-1\rangle + \langle 0,4,5\rangle$

so $\boxed{\vec{r}(x) = \langle 0,4,5\rangle + x\langle 1,-1,-1\rangle}$ is the equation of the line of intersection.

(*)

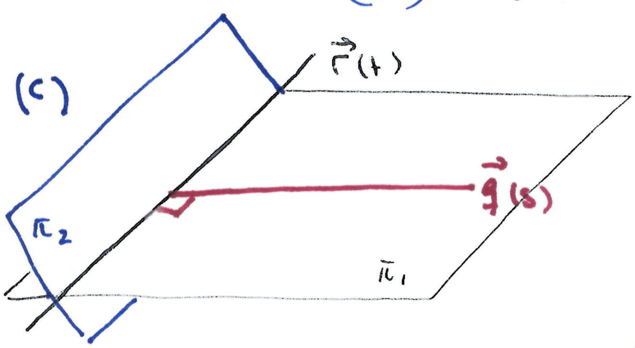
(b) By definition, the angle between two intersecting planes is the acute angle θ between their normal vectors.

$\vec{n}_1 = \langle 1, 2, -1\rangle$ $\quad$ so $\quad \vec{n}_1 \cdot \vec{n}_2 = 2+2-1 = 4-1 = 3$

$\vec{n}_2 = \langle 2, 1, 1\rangle$ $\quad |\vec{n}_1| = |\vec{n}_2| = \sqrt{1+4+1} = \sqrt{6}$

so $3 = (\sqrt{6})^2 \cos\theta = 6\cos\theta$, so $\cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta = 60^\circ}$

(c)



- $\vec{q}(s)$ lies in π_1, so it's direction $\vec{v}$ is perpendicular to $\vec{n}_1 = \langle 1,2,-1\rangle$
- $\vec{q}(s)$ is perpendicular to $\vec{r}(t)$, so their directions are perpendicular ($\vec{w} = \langle 1,-1,-1\rangle$ dir of $\vec{q}$)

So $\vec{v} = \langle a, b, c\rangle$ satisfies:

(1) $\vec{v} \cdot \vec{n} = a + 2b - c = 0$

(2) $\vec{v} \cdot \vec{w} = a - b - c = 0 \Rightarrow c = a - b$

$\Rightarrow a + 2b - (a-b) = 3b = 0$ so $b = 0$ & $c = a$.

so $\vec{v} = \langle a, 0, a\rangle = a\langle 1,0,1\rangle$ we take $a = 1$.

- $\vec{q}(s)$ has direction $\langle 1,0,1\rangle$ and passes through $(1,4,5)$

$$\boxed{\vec{q}(s) = \langle 1,4,5\rangle + s\langle 1,0,1\rangle}$$

(*) Alternative: define $\vec{r}(t)$ via a pt P & a direction $\vec{v}$ $\rightsquigarrow \boxed{\vec{r}(t) = \vec{OP} + t\langle 3,-3,-3\rangle}$

$\vec{r}(t)$ lies in π_1, so $\vec{v} \perp \vec{n}_1 = \langle 1,2,-1\rangle$

$\vec{r}(t)$ " " π_2 " $\vec{v} \perp \vec{n}_2 = \langle 2,1,1\rangle$

so $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 3,-3,-3\rangle$

pt $P = (a,b,c)$ in π_1, π_2 $\quad$ Try $a = 0 \rightarrow \left.\begin{matrix} 2b - c = 3 \\ b + c = 9 \end{matrix}\right\} \Rightarrow$ soln $b = 4$, $c = 5$ so $P = (0,4,5)$

Problem 3: From the picture we know that $\vec{PQ}$ is perpendicular to every direction in the plane π, so it's parallel to the normal direction to π, ie $\vec{n} = \langle 1,-4,2\rangle$.

Then:
- Q lies in the plane π
- Q lies in the line l passing through P and with direction $\langle 1,-4,2\rangle$.

We write the vector equation of this line: $\vec{r}(t) = \langle 3,11,6\rangle + t\langle 1,-4,2\rangle$

$= \langle 3+t, -4t+11, 2t+6\rangle$

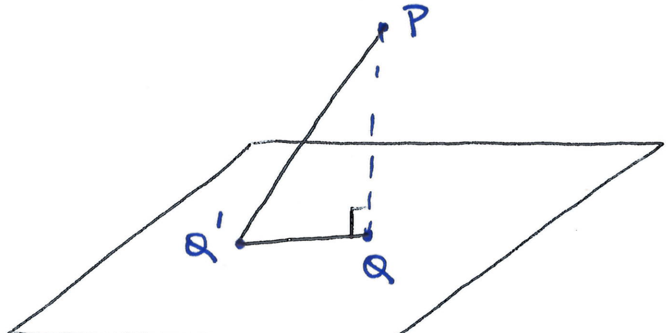
The intersection between l and π is the point Q. We plug in the components of $\vec{r}(t)$ in the equation of π and solve for t.

$$13 = (3+t) - 4(-4t+11) + 2(2t+6) = (1+16+4)t + (3-44+12)$$

$$13 = 21t + (-29) \quad \text{so} \quad 42 = 21t, \quad \boxed{t=2}$$

Q is the head of $\vec{r}(2) = \langle 5,3,10\rangle$.

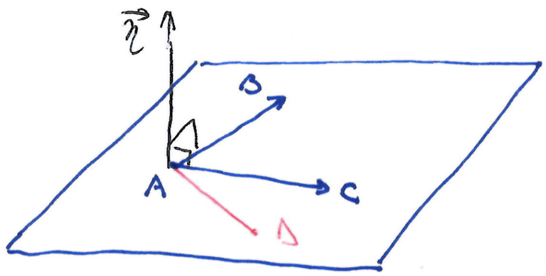
Claim: The picture is accurate, meaning Q is the closest point in π to P



$$|\vec{PQ'}|^2 = |PQ|^2 + |QQ'|^2$$

by Pythagoras Theorem, so if $Q' \neq Q$, the distance from P to Q' is larger than the distance from P to Q. In short, Q is the point of π closest to the point P.

Problem 4: To decide if these 4 pts are coplanar, it's enough to check that the 3 vectors $\vec{AB}$, $\vec{AC}$ & $\vec{AD}$ give directions generating a plane, meaning the normal direction $\vec{\eta}$ to the plane containing A, B & C <u>must</u> be perpendicular to the direction of $\vec{AD}$.



In short $(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 0$

(Recall: this formula gave the volume of the parallelopiped associated to A, B, C & D).

$\vec{AB} = \langle -2, -2, -1 \rangle$
$\vec{AC} = \langle 9, 1, 1 \rangle$
$\vec{AD} = \langle 4, -1, 1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & -1 \\ 9 & 1 & 1 \end{vmatrix} = (-1)\vec{i} - (7)\vec{j} + 16\vec{k}$$

So $(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \langle -1, -7, 16 \rangle \cdot \langle 4, -1, 1 \rangle = -4 + 7 + 16 = 19 \neq 0$

<u>Conclusion</u>: The 4 points are <u>not</u> coplanar.

Problem 5: We must find (x, y) satisfying $x^2 - 4y^2 = k$ for $k = -1, 0, 1$.

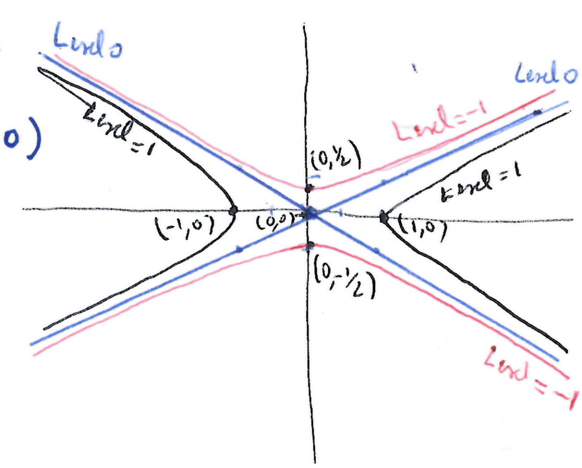
- <u>Level curve</u> (-1) $x^2 - 4y^2 = -1 \iff 4y^2 - x^2 = 1 \iff \left(\frac{y}{(1/2)}\right)^2 - x^2 = 1$

  This is a hyperbola w/ vertices $(0, \pm\frac{1}{2})$ & asymptotes $y = \pm\frac{1}{2}x$

- Level curve (0): $x^2 - 4y^2 = 0 \iff$ union of 2 lines $y = \pm\frac{1}{2}x$.

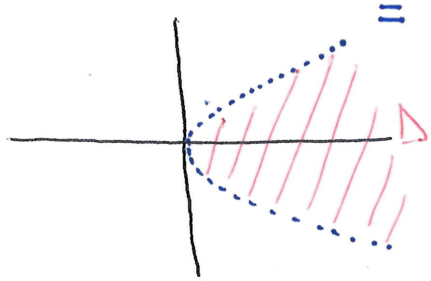
- <u>Level curve</u> (1): $x^2 - 4y^2 = 1 \iff x^2 - \left(\frac{y}{(1/2)}\right)^2 = 1$

  This is a hyperbola w/ vertices $(\pm 1, 0)$ & asymptotes $y = \pm\frac{1}{2}x$



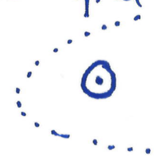
Note: The level 0 curve is the asymptote of both the level 1 & the level -1 curves.

Problem 6: (a) By definition $D = \{(x,y) \text{ in } \mathbb{R}^2 \text{ s.t. } x - y^2 > 0\}$
$= \{(x,y) \text{ in } \mathbb{R}^2 \text{ s.t. } x > y^2\}$



$x = y^2$ is a parabola

(b) For every (x,y) in D we can find a small ball around the point (x_0, y_0) that is contained in D



So all points in D are interior points

The boundary of D consist of the points on the parabola: $x = y^2$

(c) By definition D is open.

Problem 7: To prove a limit does not exist, it suffices to pick 2 paths where the limits are different.

(i) The limit does not exist.

Path 1: $x = 2y \quad y > 0$

$$\lim_{\substack{x=2y \\ y \to 0^+}} \frac{y}{\sqrt{4y^2 - y^2}} = \lim_{y \to 0^+} \frac{y}{\sqrt{3y^2}} = \lim_{y \to 0^+} \frac{y}{\sqrt{3}|y|}$$

$$= \lim_{y \to 0^+} \frac{1}{\sqrt{3}} \frac{y}{y} = \lim_{y \to 0^+} \frac{1}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$$

because $y > 0$

PATH 2: $x = 2y \quad y < 0$

$$\lim_{\substack{x=2y \\ y \to 0^+}} \frac{y}{\sqrt{4y^2 - y^2}} = \lim_{y \to 0^-} \frac{y}{\sqrt{3}|y|} = \lim_{y \to 0^-} \frac{y}{\sqrt{3}(-y)} = \boxed{\frac{-1}{\sqrt{3}}}$$

The limits along these 2 paths are different, so the limit does not exist.

(ii) The limit does not exist

Path 1: $x = y$, $x \to 0^+$

$$\lim_{\substack{x=y \\ x \to 0^+}} \frac{|x^2|}{x^2} = \lim_{x \to 0} 1 = \boxed{1}$$

Path 2: $x = -y$, $x \to 0^+$

$$\lim_{\substack{x=-y \\ x \to 0^+}} \frac{|-x^2|}{-x^2} = \lim_{x \to 0^+} \frac{x^2}{-x^2} = \lim_{x \to 0^+} -1 = \boxed{-1}$$

As with (i), the limits along these 2 paths are different, so the limit does not exist.

Problem 8: We notice that the surfaces are not in standard form, so we must first change coordinates to write them in their standard presentation.

(i) Notice the equation has a linear term in x, so we get rid of it by completing squares:

$$2 = -x^2+\frac{y^2}{4}-2x-\frac{z^2}{9} = -(x^2+2x+1)+1+\frac{y^2}{4}-\frac{z^2}{9} = -(x+1)^2+\frac{y^2}{4}-\frac{z^2}{9}+1$$

is equivalent to $1 = -(x+1)^2+\frac{y^2}{4}-\frac{z^2}{9}$ Call $x+1=u$ for our calculations

$$1 = -u^2+\frac{y^2}{4}-\frac{z^2}{9}$$ so the surface is a HYPERBOLOID OF 2 SHEETS.

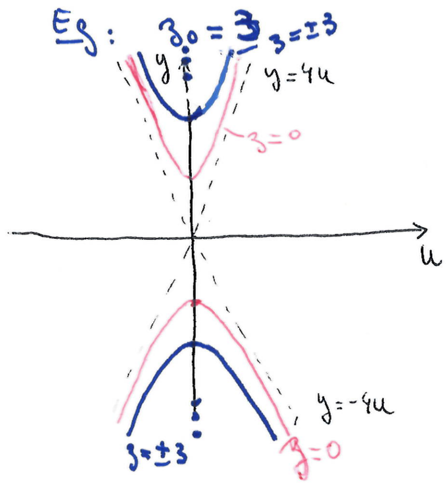
To draw it, we need to know the uy & uz-traces (yz-traces behave as the uy-traces).

uy-traces: $z=0$: $1 = -u^2+\frac{y^2}{4}$ is a hyperbola w/ asymptotes $y=\pm 4u$ & vertices $=(0,\pm 2)$

$z=\pm z_0$: $1+\frac{z_0^2}{9} = -u^2+\frac{y^2}{4}$

$$1 = \frac{y^2}{4\left(1+\frac{z_0^2}{9}\right)} - \frac{u^2}{1+\frac{z_0^2}{9}}$$ so we get a hyperbola w/ the same asymptotes but vertices further away.

Eg: $z_0=3$ $1 = \frac{y^2}{8}-\frac{u^2}{2}$ vertices $=(0,\pm 2\sqrt{2})$



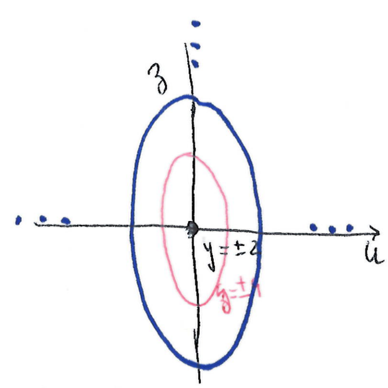
uz-traces: $y=0$ $1 = -u^2-\frac{z^2}{9}$ Empty trace

For $|y|<2$ $0<1-\frac{y^2}{4} = -u^2-\frac{z^2}{9}$ " "

$y=\pm 2$ $0 = -u^2-\frac{z^2}{9}$ $(0,0)$ = trace

$y=\pm 4$ $(|y|>2)$ $-3 = 1-\cancel{4} = -u^2-\frac{z^2}{9}$ $\Leftrightarrow$ $u^2+\frac{z^2}{9}=3$

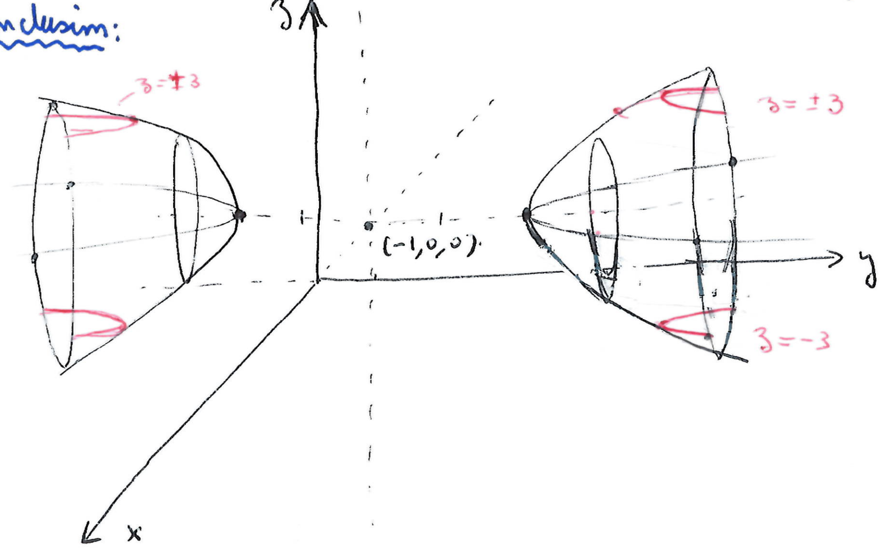
$\Leftrightarrow$ $\frac{u^2}{3}+\frac{z^2}{27}=1$ ellipse



Ellipses are concentric and grow away with $|y|$.

We remember $u = x+1$ so our standard hyperboloid of 2 sheets is <u>not</u> "centered" at $(0,0,0)$ but at $(-1,0,0)$.

Conclusion:



(ii) Here, we must make the constant term $= z$ rather than $2z$, so we divide the whole equation by 2:

$$\boxed{\frac{x^2}{4}-\frac{y^2}{16}=z}$$

, and this is the equation of the hyperbolic paraboloid, also known as a PRINGLES potato chip

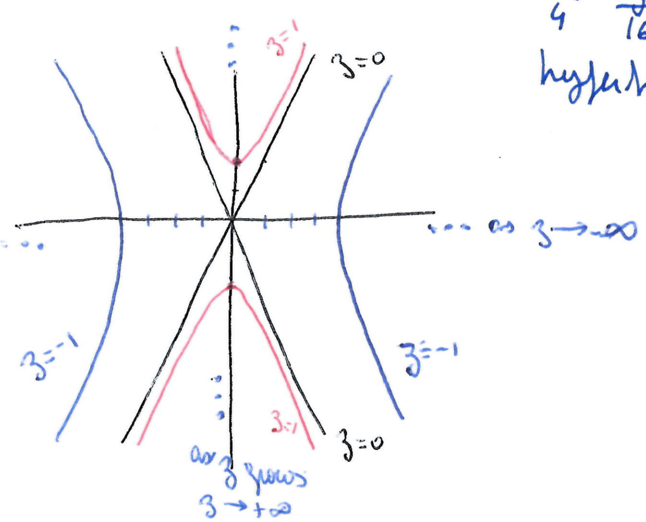
We must draw all 3 types of traces because the roles of all 3 variables is <u>not</u> symmetric:

<u>xy-traces</u>: $z=0$: $0=\frac{x^2}{4}-\frac{y^2}{16}=\left(\frac{x}{2}-\frac{y}{4}\right)\left(\frac{x}{2}+\frac{y}{4}\right)$ so union of 2 lines $y=\pm 2x$.

$z=1$: $1=\frac{x^2}{4}-\frac{y^2}{16}$ hyperbola with vertices $(\pm 2,0)$ & asymptotes $y=\pm 2x$.

$z=-1$: $-1=\frac{x^2}{4}-\frac{y^2}{16}$, equivalently: $1=\frac{y^2}{16}-\frac{x^2}{4}$

hyperbola with vertices $(0,\pm 4)$ & asymptotes $y=\pm 2x$

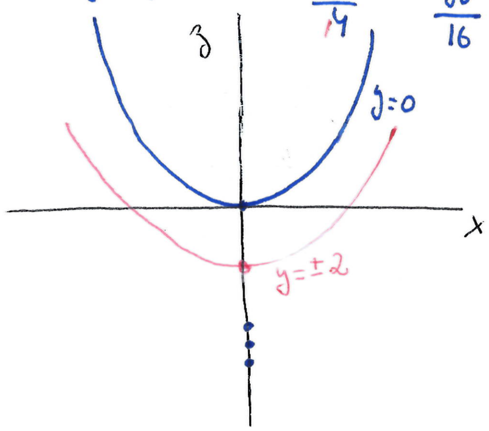


xz-Traces:

$y = 0$: $\frac{x^2}{4} = z$ parabola. with vertex $(0,0)$

$y = \pm y_0$ $\frac{x^2}{4} - \frac{y_0^2}{16} = z$ " " " $(0, -\frac{y_0}{2})$

same parabola but translated up

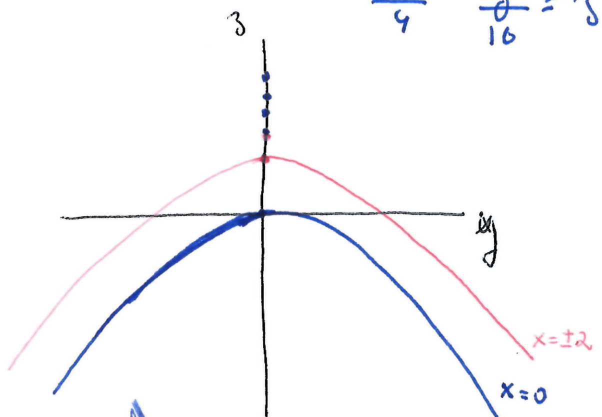


yz-Traces:

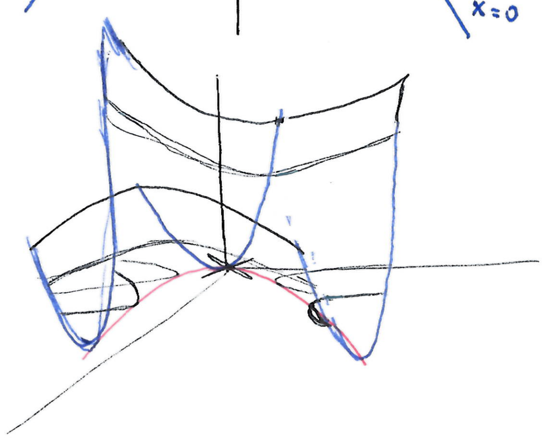
$x = 0$ $\frac{-y^2}{16} = z$ parabola with vertex $(0,0)$

$x = \pm x_0$ $\frac{x_0^2}{4} - \frac{y^2}{16} = z$ " " " $(0, \frac{x_0^2}{4})$

same parabola, but shifted up.



Conclusion:



Looks like a Pringles