MATH 2153 - Calculus III – Recitation 6

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1. Suppose that z is an implicit function of x and y satisfying $F(x, y, z) = 0$. Show that $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

2. Decide if there exists a differentiable function $f \colon \mathbb{R}^2 \to \mathbb{R}$ satisfying that $f_{xy} = x + y^2$ and $f_y = x - 1$.

3. **Derivatives of an integral.** Let $h \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $f(x, y) = \displaystyle\int_1^{xy} h(s)\, ds$. Find f_x and f_y, and decide if f is differentiable.

4. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ defined by the the formula $f(x, y) = \begin{cases} 1 & \text{if } x \leq 0 \quad \text{or} \quad x \geq y^2, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Show that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along any line through $(0, 0)$. (*Hint:* Draw the regions defining the different values of f.)

   (b) Despite part (a), show that f is discontinuous at $(0, 0)$.

   (c) Find all points of $\mathbb{R}^2$ where the function f is differentiable.

   (d) Show that all points in (c) lie in the union of two parametric curves and write down the corresponding parameterization of each of these curves.

5. **Partial derivatives and level curves.** Consider the function $z = f(x, y) = x/y^2$.

   (a) Compute the partial derivatives f_x and f_y.

   (b) Sketch the level curves for $z = -1, 0, 1$ and 2.

   (c) Compute the gradient of the function at the points $(-1, 1)$, $(0, 1)$, $(1, -1)$ and $(2, 1)$. Check that the tangent line to the level curve at each point is perpendicular to the corresponding gradient. (*Hint:* Parameterize each level curve to compute the tangent directions)

6. Consider the graph of the function $z = f(x, y) = 4 + x^2 + 3\,y^2$. Let $P_0 = (3, 4)$.

   (a) Compute the directional derivatives of f at P_0 in the directions $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ and $\mathbf{v} = \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle$, and decide if the function is increasing or decreasing when moving in the directions $\mathbf{u}$ and $\mathbf{v}$.

   (b) If we place a ball at the point $(3, 4, 61)$, find the curve that describes the trajectory of the ball when moving from this point to the vertex $(0, 0, 4)$ of the paraboloid. (*Hint:* The ball will move along the path of steepest descent. A picture might help visualize things.)

7. **Fluid flow.** A basin of circulating water is represented by the square region $R = \{(x, y) : 0 \le x, y \le 1\}$. The velocity components of the water at position (x, y) in R are

$$\begin{cases} \text{the east-west velocity} & u(x, y) = x(1 - x)(1 - 2y), \\ \text{the north-south velocity} & v(x, y) = y(y - 1)(1 - 2x). \end{cases}$$

   These velocity components produce the flow patterns in the picture below. Find the rates of change of the water speed in the x- and y-directions. (*Hint:* Use the chain rule and implicit differentiation.)