Problem 3: We use the Fundamental Theorem of Calculus & the definition of antiderivative.

\[ g(x) = f(x, b) = \int_{a}^{x} h(s) \, ds = \int_{b}^{t(x)} \dot{h}(s) \, ds = \frac{d}{dx} \int_{a}^{t(x)} h(s) \, ds = H(t(x))' \]

so \[ g'(x) = H'(t(x)) \cdot t'(x) = h(bx) \cdot b. \]

\[ \Rightarrow f_x(a, b) = g'(a) = h(ab) \cdot b. \]

By symmetry, \[ g(y) = f(a, y) = \int_{a}^{y} h(s) \, ds = \int_{b}^{t(y)} \dot{h}(s) \, ds = H(t(y)) \]

where \( H \) is the antiderivative of \( h \).

so \[ g'(y) = f_y(a, y) = H'(t(y)) \cdot t'(y) = h(t(y)) \cdot a = h(ay) \cdot a \]

\[ \Rightarrow f_y(a, b) = g'(b) = h(ab) \cdot a. \]

Problem 2: Answer: No.

We differentiate \( f_y \) with respect to \( x \): \( (f_y)_x = 1 \)

so \( f_{xy} = x + y^2 \) & hence \( f_{yx} = 1 \).

The mixed derivatives are continuous, and distinct, which contradicts the mixed partials. Thus, \( f \) cannot exist.

We conclude from this that \( f \) cannot exist.

Note: We can try to find \( f \) by integrating with respect to one of the variables. Partially:

\[ f(x, y) = \int f_y \, dy + C \]

where \( C \) is a constant function in the \( y \) variable, so \( C = C(x) \) depends only on \( x \).

\[ f(x, y) = \int (x-y) \, dy + C(x) = (x-y) y + C(x). \]

Then \( f_x = y + C'(x) \) and \( (f_x)_y = 1 \neq x + y^2 \), so \( f \) cannot exist.
Problem 1: In this exercise, we want to use implicit differentiation.

By the statement, we know that \( g = g(x, y) \).

So \( g(x, y) = F(x, y; 3(x, y)) = 0 \).

We draw the tree dependence and note that:

- \( \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0 \) because \( x, y \) are independent variables.
- \( \frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1 \).

By the Chain Rule, we sum the contribution of each branch:

\[
0 = 6_x = F_x(x, y, 3) \cdot 1 + F_y(x, y, 3) \cdot 0 + F_3(x, y, 3) \cdot 3x.
\]

\( 6 \) is constant:

\[
0 = 6_y = F_x(x, y, 3) \cdot 0 + F_y(x, y, 3) \cdot 1 + F_3(x, y, 3) \cdot 3y.
\]

So, provided \( F_3(x, y, 3) \neq 0 \), we have:

\[
\frac{\partial^2 g}{\partial x \partial y}(x, y) = \frac{-F_x(x, y, 3)}{F_3(x, y, 3)}, \quad \frac{\partial^2 g}{\partial y \partial x}(x, y) = \frac{-F_y(x, y, 3)}{F_3(x, y, 3)}.
\]

Problem 4: (a) We start by drawing the regions of \( \mathbb{R}^2 \) where \( t \) has constant value \( =0 \) or \( =1 \).

To compute the limit, we parametrize the line:

- Vertical line \( \vec{r}(t) = <0, t> \)

\[
\lim_{(x, y) \to (0, t)} f(x, y) = \lim_{t \to 0} f(0, t) = \lim_{t \to 0} 1 = 1.
\]
All other lines $\mathbb{R}(t) = \langle t, mt \rangle$ ($y = mx$ is the defining equation)

\[
\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{t \to 0} f(t, mt)
\]

To know the value of $f(t, mt)$, we have to determine in which region the point $(t, mt)$ is:

- If $t < 0$, $(t, mt)$ lies to the left of the y-axis & so \[f(t, mt) = 1, \quad \text{so} \quad \lim_{t \to 0^-} f(t, mt) = 1\]

- If $t > 0$, $(t, mt)$ lies to the right of the y-axis. We need to check if it lies above or below the parabola when $t$ is very small.

$m > 0$

\[
\frac{t^2}{mt} = \left(\frac{t}{m}, t_0\right)
\]

$m < 0$

We find the intersection $t_0$ of the line & the parabola. $(t_0^2, t_0) = (t, mt)$

\[
\begin{align*}
m &\geq 0 \\
mt = mt_0 &= t_0 \quad \Rightarrow \quad m = \frac{1}{t_0} \quad \Rightarrow \quad t_0 = \frac{1}{m} \\
\text{when } 0 < t < \frac{1}{m}, \text{ the line is below the parabola and so } & f(t, mt) = 1 \quad \lim_{t \to 0^+} 1
\end{align*}
\]

\[
\begin{align*}
m &< 0 \\
t_0 = x \quad \Rightarrow \quad -t_0 = mt = mt_0^2 \\
\text{so } t_0 &= \frac{1}{m} > 0. \quad \text{When, } 0 < t_0 < \frac{1}{m}, \text{ the line is above the parabola and so } & f(t, mt) = 1 \quad \lim_{t \to 0^+} 1
\end{align*}
\]

Conclusion: \[
\lim_{(x,y) \to (0,0)} f(x,y) = 1
\]

When $(x, y)$ along any line through $(0,0)$
(b) To show \( f \) is discontinuous at \((0, 0)\) we must find a path along which the limit is not 1. The path has to be in the region where the function has value 0.

\[
\text{Eg: } x = \frac{1}{2} y^2
\]

\[
\lim_{(x,y) \to (0,0)} f(x, y) = 0.
\]

By the Path Test, the limit does not exist, so \( f \) is discontinuous at \((0, 0)\).

(c) We know if \( f \) is discontinuous, it cannot be differentiable. \( f \) is discontinuous at the lines \((x = 0)\) and at all points in the parabola \((x = y^2)\).

For the line \( x = 0 \):

\[
\lim_{x \to 0^-} f(x, y_0) = \lim_{x \to 0^+} f(x, y_0) = 0
\]

By the Path Test, the function is discontinuous at \((0, y_0)\).

At the point \((0, y_0)\), we know \( f \) is discontinuous by (b).

For the parabola, pick vertical lines \((x = x_0)\), \( x_0 > 0 \).

At the point \((x_0, \Gamma x_0)\):

\[
\lim_{y \to \Gamma x_0^-} f(x_0, y) = 1 = \lim_{y \to \Gamma x_0^-}
\]

\[
\lim_{y \to \Gamma x_0^+} f(x_0, y) = 0
\]

Again, by the path test, \( f \) is discontinuous at \((x_0, \Gamma x_0)\) and at \((x_0, -\Gamma x_0)\).
At all other pts, $f$ is locally constant so it's differentiable ($\nabla f \neq \nabla \delta$, so $h_x, h_y$ continuous).

2. $f = 0$ around the points $P_1, P_2$

(d) The curves are the line $x = 0$, parameterized as $\vec{r}(t) = \langle 0, t \rangle \text{ for } t \in \mathbb{R}$.

the parabola $x = y^2$, $\vec{r}(t) = \langle t^2, t \rangle$ for $t \in \mathbb{R}$

Problem 5: (a) We use the derivative rules:

(a) $h_x = \frac{1}{y^2}$ (y is constant)

$\frac{dy}{dx} = \frac{2x}{y^3}$ (x is constant)

(b) We draw the 4 curves $\frac{x}{y^2} = 3$ so $x = 30y^2$ where $3 = -\frac{1}{9}, -\frac{1}{9}, 0, 3$.

(c) At the level curve $x = 30y^2$, the gradient equals:

$\nabla f(x, y) = \langle \frac{1}{y^2}, \frac{-2x}{y^3} \rangle = \langle \frac{1}{y^2}, \frac{-2 \cdot 30y^2}{y^3} \rangle = \langle \frac{1}{y^2}, \frac{-2 \cdot 30}{y} \rangle$

Eq: $(1, 1)$ lies in $x = y^2 \implies \nabla f(1, 1) = \langle 1, 0 \rangle$

$\nabla f(0, 1) = \langle 0, 1 \rangle$

$\nabla f(-1, 1) = \langle 0, 1 \rangle$

$\nabla f(2, 1) = \langle 1, -4 \rangle$

To check that the gradient is perpendicular to the tangent direction, we
Parameterize the curve and check that $\nabla f(x(t), y(t)) \cdot \vec{r}'(t) = 0$.

- Level curve $z_0 = 0$: tangent direction = $<0, 1>$ because $\vec{r}(t) = <0, t>$.
  \[ \nabla f(0, y) = <\frac{1}{y^2}, 0> \perp <0, 1> \checkmark \]

- Level curve $z_0 = 1$: $\vec{r}(t) = <t^2, t> \Rightarrow \vec{r}'(t) = <2t, 1>$.
  \[ \nabla f(t^2, t) = <\frac{1}{t^2}, -\frac{2}{t^2}> \perp <2t, 1> \checkmark \]

- Level curve $z_0 = -1$: $\vec{r}(t) = <-t^2, t> \Rightarrow \vec{r}'(t) = <-2t, 1>$.
  \[ \nabla f(-t^2, t) = <\frac{1}{t^2}, \frac{2}{t^2}> \perp <-2t, 1> \checkmark \]

- Level curve $z_0 = 2$: $\vec{r}(t) = <2t^2, t> \Rightarrow \vec{r}'(t) = <4t, 1>$.
  \[ \nabla f(2t^2, t) = <\frac{1}{t^2}, -\frac{4}{t^2}> \perp <4t, 1> \checkmark \]

**Problem 6:** We start by drawing the surface. Elliptic paraboloid.

(a) We start by computing the gradient:

\[ \nabla f(x, y) = <f_x(x, y), f_y(x, y)> \]
\[ = <2x, 6y> \Rightarrow \nabla f(3, 4) = <6, 24> \]
\[ D_u f(3, 4) = <6, 24> \cdot \vec{u} = \frac{6 + 24}{12} = \frac{30}{12} > 0 \]
\[ D_{\vec{w}} f(3, 4) = <6, 24> \cdot \vec{w} = 3 - 12\sqrt{3} < 0 \]

Since $D_u f(3, 4) > 0$, the function $f$ is increasing when moving in the direction of $\vec{u}$.

Since $D_{\vec{w}} f(3, 4) < 0$, the function $f$ is decreasing when moving in the direction of $\vec{w}$.

(b) We write $\vec{r}(x) = <x, y(x)>$ and $\vec{r}(3) = <3, 4>$ so $y(3) = 4$. 
The path \( r_{th} \) will have steepest descent if at every point \((x,y)\) we move in the direction 
\[
\frac{-\nabla f(x,y)}{1/\nabla f(x,y)} \]
This direction is also given by the tangent direction \(-\vec{t}(t)\), which means that
\[
\frac{y'(x)}{x} = \frac{fy(x,y)}{fx(x,y)} = \frac{6y}{2x} \quad (the \ slope \ is \ the \ same!)
\]
So the function \( y = y(x) \) we are after satisfies:
\[
\begin{align*}
y'(x) &= \frac{3y}{x} \\
y(3) &= 4 \quad \text{(starting pt: } (3,4) \text{)}
\end{align*}
\]
This defines a differential equation, which we can solve.

\[
\begin{align*}
(\ln y)' &= \frac{y'(x)}{y} = \frac{3}{x} & \Rightarrow \text{ integrate } \ln y = \int \frac{3}{x} \, dx + C \\
&= 3 \ln x + C
\end{align*}
\]
We exponentiate \( y = e^{3 \ln x + C} = (e^C) x^3 \) \( \quad \text{constant} = a \)

Use use the initial condition to find \( a \).
\[
y = y(3) = a \cdot 27 \quad \Rightarrow \quad a = \frac{4}{27}
\]
is the desired curve.

What's the end point? \( y(0) = 0 \). \( \Rightarrow \) The end point is the vertex of the paraboloid.

**Problem 7:** The water determining stream lines with constant speed.

By definition \( S(x,y) = \sqrt{u^2(x,y) + v^2(x,y)} \) at position \((x,y)\),

We write \( S = S(u(x,y), v(x,y)) \) & use the chain rule.
\[
\begin{align*}
\frac{ds}{dx} &= Su \cdot \frac{\partial u}{\partial x} + Sv \cdot \frac{\partial v}{\partial x} \\
\frac{ds}{dy} &= Su \cdot \frac{\partial u}{\partial y} + Sv \cdot \frac{\partial v}{\partial y}
\end{align*}
\]
To find $s_u$ & $s_v$, it's easier to use $s^2(u,v) = u^2 + v^2$
we do implicit differentiation with respect to $u$ & $v$, use the chain rule
\[
\begin{align*}
\frac{d}{du} (s^2 = u^2 + v^2) & \quad \text{gives} \quad 2s \cdot s_u = 2u \implies s_u = \frac{u}{s} \\
\frac{d}{dv} (s^2 = u^2 + v^2) & \quad \text{gives} \quad 2s \cdot s_v = 2v \implies s_v = \frac{v}{s}
\end{align*}
\]
provided $s \neq 0$. (need is not $=0$).

we replace these values back in $x$
\[
\frac{ds}{dx} = \frac{u}{s} \frac{du}{dx} + \frac{v}{s} \frac{dv}{dx} \quad \text{&} \quad \frac{ds}{dy} = \frac{u}{s} \frac{du}{dy} + \frac{v}{s} \frac{dv}{dy}
\]

In our case:
\[
\begin{align*}
\frac{du}{dx} &= (1-2y)(1-2x) \\
\frac{du}{dy} &= -2x(1-x)
\end{align*}
\]
\[
\begin{align*}
\frac{dv}{dx} &= -2y(y-1) \\
\frac{dv}{dy} &= (2y-1)(1-2x)
\end{align*}
\]
\[
s = \sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}
\]

Conclusion:
\[
\frac{ds}{dx} = \frac{x(1-x)(1-2y)^2(1-2x) - 2y(y-1)^2(1-2x)}{\sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}}
\]
\[
\frac{ds}{dy} = \frac{-2x^2(1-x)^2(1-2y) + y(y-1)(1-2x)(2y-1)}{\sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}}
\]