Recitation VI (2/25/16)

Problem 3: We use the Fundamental Theorem of Calculus & the definition of f_x, f_y

$H(a,b)$: $g(x) = f(x,b) = \int_1^{bx} h(s)\,ds = \int_1^{t(x)} h(s)\,ds = \underbrace{H}_{\text{antiderivative}}(t(x))$.

So $g'(x) = H'(t(x))\,t'(x) = h(bx)\,b$.

$\Rightarrow f_x(a,b) = g'(a) = h(ab)\,b$.

By symmetry: $g(y) = f(a,y) = \int_1^{ay} h(s)\,ds = \int_1^{t(y)} h(s)\,ds = H(t(y))$

where H is the antiderivative of h.

So $g'(y) = f_y(a,y) = H'(t(y))\,t'(y) = h(t(y))\,a = h(ay)\,a$

$\Rightarrow f_y(a,b) = g'(b) = h(ab)\,a$.

Problem 2: ANSWER: <u>NO</u>.

We differentiate f_y with respect to x: $(f_y)_x = 1$

So $f_{xy} = x + y^2$ continuous
$f_{yx} = 1$ "

} The mixed derivatives are continuous and distinct, which contradicts the Mixed Partials Thm.

We conclude from this that f cannot exist.

<u>Note</u>: We can try to find f by integrating partials with respect to one of the variables.

$f_{(x,y)} = \int f_y\,dy + C$, where C is a constant function in the y variable, so $C = C(x)$ depends ONLY on x

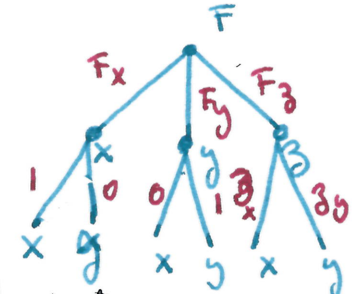
$f(x,y) = \int (x-1)\,dy + C(x) = (x-1)y + C(x)$.

Then $f_x = y + C'(x)$ & $(f_x)_y = 1 \neq x+y^2$ so f cannot exist!

Problem 1: In this exercise, we want to use implicit differentiation.

By the statement, we know that $z = z(x,y)$.

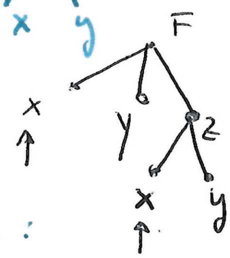
So $G(x,y) = F(x, y, z(x,y)) = 0$



We draw the tree dependence and note that

- $\dfrac{\partial x}{\partial y} = \dfrac{\partial y}{\partial x} = 0$ because x & y are INDEPENDENT variables
- $\dfrac{\partial x}{\partial x} = \dfrac{\partial y}{\partial y} = 1$

so the tree is



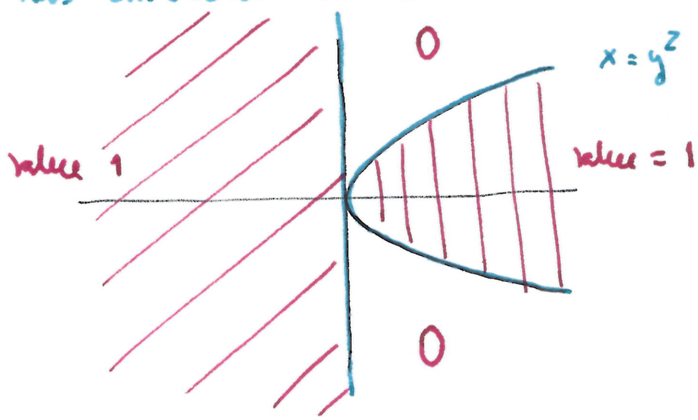
By the Chain Rule, we sum the contribution of each branch:

$$\begin{cases} 0 = G_x = F_x(x,y,z)\, 1 + F_y(x,y,z)\, 0 + F_z(x,y,z)\, z_x . \\ \quad G \text{ is constant} \\ 0 = G_y = F_x(x,y,z) \cdot 0 + F_y(x,y,z)\, 1 + F_z(x,y,z)\, z_y \end{cases}$$

So, provided $F_z(x,y,z) \neq 0$, we have

$$\frac{\partial z}{\partial x}(x,y) = \frac{-F_x(x,y,z)}{F_z(x,y,z)} \quad ; \quad \frac{\partial z}{\partial y}(x,y) = \frac{-F_y(x,y,z)}{F_z(x,y,z)} .$$

Problem 4: (a) We start by drawing the regions of $\mathbb{R}^2$ where f has constant value $= 0$ or $= 1$.



To compute the limit, we parametrize the lines.

- Vertical line $\vec{r}(t) = \langle 0, t \rangle$

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y) \text{ in } (x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0} 1 = 1$$

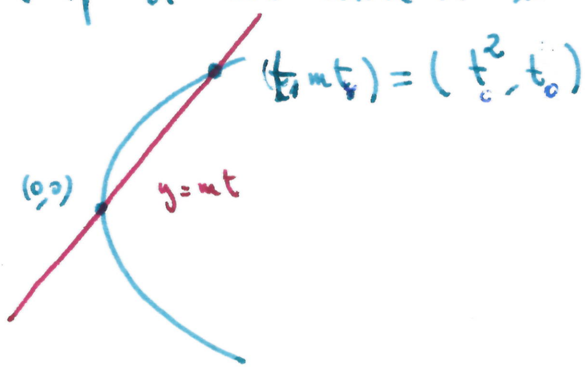
All other lines $\vec{r}(t) = \langle t, mt \rangle$ ($y = mx$ is the defining equation)

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y) \text{ along } y=mx}} f(x,y) = \lim_{t\to 0} f(t,mt)$$

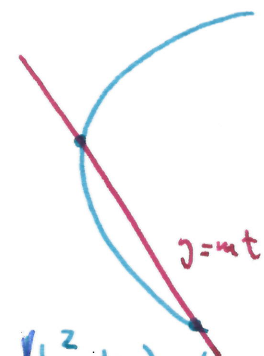
To know the value of $f(t,mt)$, we have to determine in which region the point (t, mt) is:

- If $t<0$, (t,mt) lies to the ~~right~~ left of the y-axis & so $f(t,mt) = 1$, so $\lim\limits_{t\to 0^-} f(t,mt) = 1$
- If $t>0$, (t,mt) lies to the right of the y axis. We need to check if it lies above or below the parabola when t is very small.

m>0



m<0



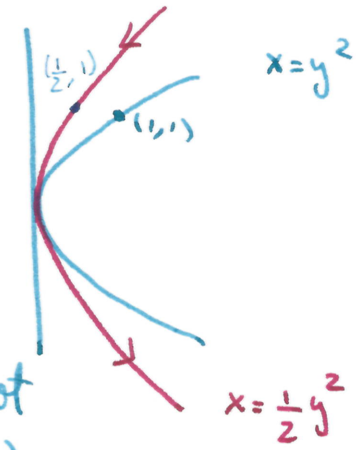
We find the intersection pt of the line & the parabola. $(t_0^2, -t_0) = (t, mt)$

m>0 $\left.\begin{array}{l} t = t_0^2 \\ mt = mt_0^2 = t_0 \end{array}\right\}$ $m = \frac{1}{t_0}$ or $\boxed{t_0 = \frac{1}{m}}$ when $0<t<\frac{1}{m}$, the line is below the parabola and so $f(t,mt) = 1 \xrightarrow[t\to 0^+]{} 1$.

m<0 $\left.\begin{array}{l} t_0^2 = t \\ -t_0 = mt = mt_0^2 \end{array}\right\}$ so $t_0 = \frac{-1}{m} > 0$. When, $0 < t_0 < \frac{-1}{m}$, the line is above the parabola and so $f(t,mt) = 1 \xrightarrow[t\to 0^+]{} 1$

Conclusion $\lim\limits_{\substack{(x,y)\to(0,0) \\ (x,y) \text{ along ANY line through } (0,0)}} f(x,y) = 1$

(b) To show f is discontinuous at $(0,0)$ we must find a path along which the limit is <u>NOT</u> 1. The path has to be in the region where the function has value $=0$.



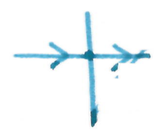
Eg: $\quad x = \frac{1}{2}y^2$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=\frac{1}{2}y^2}} f(x,y) = 0.$$

By the Path Test, the limit does not exist, so f is discontinuous at $(0,0)$.

(c) We know if f is discontinuous it cannot be differentiable. f is discontinuous at the lines $(x=0)$ & at all points in the parabola $(x=y^2)$.

. For the line $x=0$



we pick horizontal lines $y=y_0\neq 0$

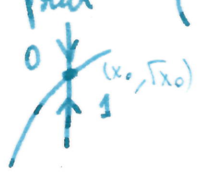
$$\begin{cases} \lim\limits_{x\to 0^-} f(x,y_0) = \lim\limits_{x\to 0^-} 1 = 1 \\ \lim\limits_{x\to 0^+} f(x,y_0) = \lim\limits_{x\to 0^+} 0 = 0 \end{cases}$$

$\leadsto$ by the Path Test, the function is discontinuous at $(0,y_0)$.

• At the point $(0,0)$, we know f is discontinuous by (b).

. For the parabola, pick vertical lines $(x=x_0)$. $\quad x_0>0$.

At the point $(x_0,\sqrt{x_0})$:



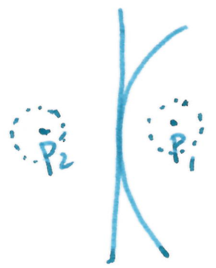
$$\begin{cases} \lim\limits_{y\to\sqrt{x_0}^-} f(x_0,y) = \lim\limits_{y\to\sqrt{x_0}^-} 1 = 1 \\ \lim\limits_{y\to\sqrt{x_0}^+} f(x_0,y) = \lim\limits_{y\to\sqrt{x_0}^+} 0 = 0 \end{cases}$$

At the point $(x_0,-\sqrt{x_0})$

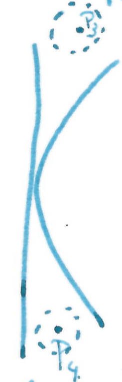
$$\begin{cases} \lim\limits_{y\to-\sqrt{x_0}^+} f(x_0,y) = \lim\limits_{y\to-\sqrt{x_0}^+} 1 = 1 \\ \lim\limits_{y\to-\sqrt{x_0}^-} f(x_0,y) = \lim\limits_{y\to} 0 = 0 \end{cases}$$

Again, by the path test, f is discontinuous at $(x_0,\sqrt{x_0})$ & at $(x_0,-\sqrt{x_0})$ for $x_0>0$.

At all other pts, f is locally constant so it's differentiable ($\nabla f \equiv \vec{0}$, so f_x, f_y continuous)



$f \equiv b$ around the 2 points P_1 & P_2



$f \equiv 0$ around P_3 & P_4

(d) The curves are → the line $x=0$, parameterized as $\vec{r}(t) = \langle 0, t \rangle$ for t in $\mathbb{R}$
→ the parabola $x = y^2$, " " $\vec{r}(t) = \langle t^2, t \rangle$ for t in $\mathbb{R}$.

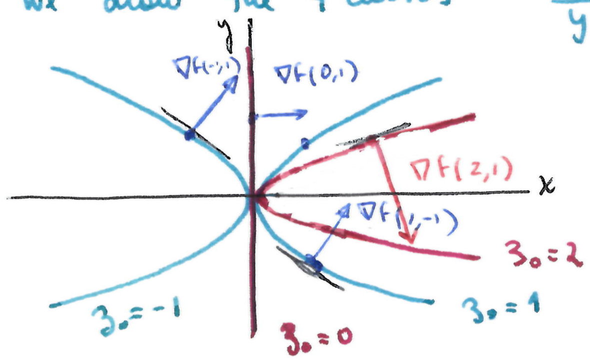
Problem 5:

(a) We use the derivation rules:

(a) $f_x = \dfrac{1}{y^2}$ (y is constant)

$f_y = \dfrac{-2x}{y^3}$ (x is constant)

(b) We draw the 4 curves $\dfrac{x}{y^2} = z_0 \rightsquigarrow x = z_0 y^2$ where $z_0 = -1, 0, 1, 2$.



(c) At the level curve $x = z_0 y^2$, the gradient equals:

$$\nabla f_{(x,y)} = \left\langle \frac{1}{y^2}, \frac{-2x}{y^3} \right\rangle = \left\langle \frac{1}{y^2}, \frac{-2 z_0 y^2}{y^3} \right\rangle = \left\langle \frac{1}{y^2}, \frac{-2 z_0}{y} \right\rangle$$

Eg:
- $(1,-1)$ lies in $x = y^2 \rightsquigarrow \nabla f(1,-1) = \langle 1, +2 \rangle$
- $(0,1)$ " " $x = 0 \rightsquigarrow \nabla f(0,1) = \langle 1, 0 \rangle$
- $(-1,1)$ " " $x = -y^2 \rightsquigarrow \nabla f(-1,1) = \langle 1, 2 \rangle$
- $(2,1)$ " " $x = 2y^2 \rightsquigarrow \nabla f(2,1) = \langle 1, -4 \rangle$

To check that the gradient is perpendicular to the tangent directions, we parameterize the curves & check that $\nabla f(x(t),y(t)) \cdot \vec{r}\,'(t) = 0$.

- Level curve $z_0=0$: tangent direction $=\langle 0,1\rangle$ because $\vec{r}(t)=\langle 0,t\rangle$.

  and $\nabla f(0,y) = \langle \frac{1}{y^2}, 0\rangle \perp \langle 0,1\rangle$ ✓

- Level curve $z_0=1$: $\vec{r}(t) = \langle t^2, t\rangle \rightsquigarrow \vec{r}\,'(t) = \langle 2t, 1\rangle$.

  $\nabla f(t^2,t) = \langle \frac{1}{t^2}, \frac{-2}{t}\rangle \perp \langle 2t,1\rangle$ ✓

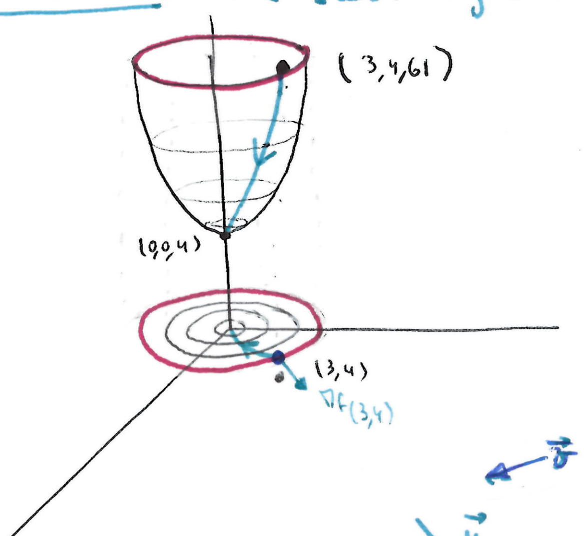
- Level curve $z_0=-1$: $\vec{r}(t) = \langle -t^2, t\rangle \rightsquigarrow \vec{r}\,'(t) = \langle -2t, 1\rangle$

  $\nabla f(-t^2,t) = \langle \frac{1}{t^2}, \frac{2}{t}\rangle \perp \langle -2t,1\rangle$ ✓

- Level curve $z_0=2$: $\vec{r}(t) = \langle 2t^2, t\rangle \rightsquigarrow \vec{r}\,'(t) = \langle 4t, 1\rangle$

  $\nabla f(2t^2,t) = \langle \frac{1}{t^2}, \frac{-4}{t}\rangle \perp \langle 4t,1\rangle$ ✓

Problem 6: We start by drawing the surface (elliptic paraboloid)



(a) We start by computing the gradient:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

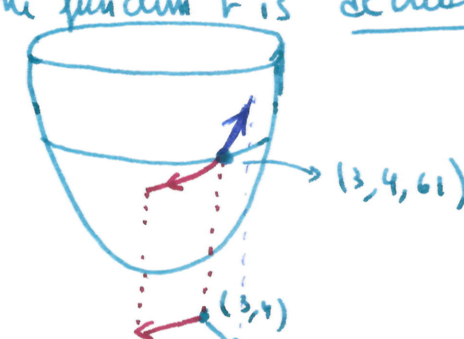
$$= \langle 2x, 6y\rangle. \rightsquigarrow \nabla f(3,4) = \langle 6,24\rangle$$

$$D_{\vec{u}} f(3,4) = \langle 6,24\rangle \cdot \vec{u} = \frac{6+24}{\sqrt{2}} = \frac{30}{\sqrt{2}} > 0$$

$$D_{\vec{v}} f(3,4) = \langle 6,24\rangle \cdot \vec{v} = 3-12\sqrt{3} < 0$$

- Since $D_{\vec{u}} f(3,4) > 0$, the function f is increasing when moving in the direction of $\vec{u}$

- Since $D_{\vec{v}} f(3,4) < 0$, the function f is decreasing when moving in the direction of $\vec{v}$.



(b) We write $\vec{r}(x) = \langle x, y(x)\rangle$ and $\vec{r}(3) = \langle 3,4\rangle$ so $y(3)=4$.

The path $\vec{r}_{(t)}$ will have steepest descent if at every pt (x,y) we move in the direction $= \dfrac{-\nabla f(x,y)}{|\nabla f(x,y)|}$.

This direction is also given by the opposite tangent direction $-\vec{r}\,'(t)$, which means that $-\langle 1, y'(x)\rangle$

$$\frac{y'(x)}{1} = \frac{f_y(x,y)}{f_x(x,y)} = \frac{6y}{2x} \quad \text{(the slope is the same!)}$$

So the function $y = y(x)$ we are after satisfies:

$$\begin{cases} y'(x) = \dfrac{3y}{x} \\ y(3) = 4 \qquad \text{(starting pt} = (3,4)) \end{cases}$$

This defines a differential equation, which we can solve.

$$(\ln y)' = \frac{y'(x)}{y} = \frac{3}{x} \implies \text{integrate} \quad \ln y = \int \frac{3}{x}\,dx + C$$
$$= 3\ln x + C$$

We exponentiate $\implies y_{(x)} = e^{3\ln x + C} = \underbrace{(e^C)}_{\text{constant}=a} x^3$

Use the initial condition to find a.

$$4 = y(3) = a\,27 \implies a = \frac{4}{27}, \text{ so } \boxed{y(x) = \frac{4}{27}x^3}$$

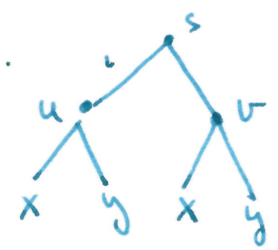
is the desired curve.

What's the end pt? $y(0) = 0$. $\Rightarrow$ The end point is the vertex of the paraboloid.

Problem 7: The water determines stream-lines with constant speed.

By definition $s(x,y) = \sqrt{u^2(x,y) + v^2(x,y)}$ at position (x,y).

We write $s = s(u(x,y), v(x,y))$. & use the chain rule.



$$(*)\begin{cases} \dfrac{\partial s}{\partial x} = s_u \cdot \dfrac{\partial u}{\partial x} + s_v \dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial s}{\partial y} = s_u \cdot \dfrac{\partial u}{\partial y} + s_v \dfrac{\partial v}{\partial y}. \end{cases}$$

To find s_u & s_v, it's easier to use $s^2_{(u,v)} = u^2 + v^2$

We do implicit differentiation with respect to u & v & use the chain rule

$$\begin{cases} \dfrac{\partial}{\partial u}\left(s^2 = u^2 + v^2 \right) & \text{gives} \quad 2s \cdot s_u = 2u \Rightarrow s_u = \dfrac{u}{s} \\[2ex] \dfrac{\partial}{\partial v}\left(s^2 = u^2 + v^2 \right) & \text{gives} \quad 2s\, s_v = 2v \Rightarrow s_v = \dfrac{v}{s} \end{cases}$$

provided $s \neq 0$. (speed is <u>not</u> $= 0$).

We replace these values back in (*)

$$\frac{\partial s}{\partial x} = \frac{u}{s}\frac{\partial u}{\partial x} + \frac{v}{s}\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial s}{\partial y} = \frac{u}{s}\frac{\partial u}{\partial y} + \frac{v}{s}\frac{\partial v}{\partial y}$$

In our case: $\begin{cases} \dfrac{\partial u}{\partial x} = (1-2y)(1-2x) \\[2ex] \dfrac{\partial u}{\partial y} = -2x(1-x) \end{cases}$ $\quad \begin{cases} \dfrac{\partial v}{\partial x} = -2y(y-1) \\[2ex] \dfrac{\partial v}{\partial y} = (2y-1)(1-2x) \end{cases}$

$$s = \sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}$$

<u>Conclusion:</u>

$$\frac{\partial s}{\partial x} = \frac{x(1-x)(1-2y)^2(1-2x) - 2y^2(y-1)^2(1-2x)}{\sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}}$$

$$\frac{\partial s}{\partial y} = \frac{-2x^2(1-x)^2(1-2y) + y(y-1)(1-2x)^2(2y-1)}{\sqrt{x^2(1-x)^2(1-2y)^2 + y^2(y-1)^2(1-2x)^2}}.$$