Recitation VII (03/03/16)

Problem 1: We use linear approximation:

$$L_{(x,y,z)} = f_{(2,1,1)} + \frac{\partial f}{\partial x}(2,1,1)\,(x-2) + \frac{\partial f}{\partial y}(2,1,1)\,(y-1) + \frac{\partial f}{\partial z}(2,1,1)(z-1)$$

is a good approx of $f(x,y,z)$ for (x,y,z) near $(2,1,1)$

In our case : $x = 2.1$, $y = 0.9$, $z = 1.2$.

$$L(2.1, 0.9, 1.2) = 4 + (-3)(0.1) + 2(-0.1) + 6(0.2) = \boxed{4.7}$$

Problem 2: The normal direction to the tangent plane is $\vec{n} = \langle -f_x(1,2)\ , -f_y(1,2)\ , 1\rangle$

To compute ∇f, we rewrite $f_{(x,y)} = y^x = e^{x\ln y}$ & use the derivation rules :

$$\begin{cases} f_x = e^{x\ln y}\cdot \ln y = y^x \ln y & \rightsquigarrow f_x(1,2) = 2^1 \ln 2 = 2\ln 2 = \ln 4 \\ f_y = e^{x\ln y}\cdot \dfrac{x}{y} = y^{x-1}x . & \rightsquigarrow f_y(1,2) = 2^{1-1}\cdot 1 = 1 \end{cases}$$

The tangent plane passes through the point $(1, 2, f_{(1,2)}) = (1,2,2)$, so its equation is

$$\vec{n}\cdot\langle x,y,z\rangle = \vec{n}\cdot\langle 1,2,2\rangle$$

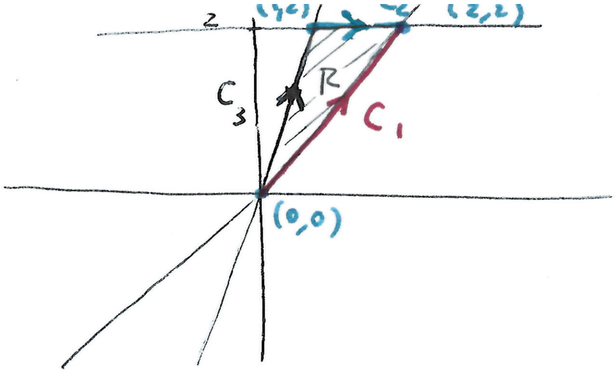
$$-(\ln 4)\,x - y + z = -\ln 4 - 2 + 2 = -\ln 4$$

$$\boxed{z = (\ln 4)\,x + y - \ln 4}$$

Problem 3: This problem involves computing critical pt & local extrema & absolute extrema of a differentiable function $f_{(x,y)}$ on a region R.

The partials of f of <u>any</u> order are differentiable (because f is a rational function whose denominator never vanishes on $\mathbb{R}^2$) so all mixed partials agree. In addition local extrema will be critical pts where $\nabla f_{(x,y)} = \vec{0}$ & we can potentially use the Second Derivative test to decide the nature of the critical pts.

(1) As suggested by the hint, we start by drawing the region R. We will later parametrize the boundary to find extremal values of f on it.



C_1 : $\vec{r}_1(t) : [0,2] \longrightarrow \mathbb{R}^2 \quad \vec{r}_1(t) = \langle t, t \rangle$

C_2 : $\vec{r}_2(t) : [1,2] \longrightarrow \mathbb{R}^2 \quad \vec{r}_2(t) = \langle t, 2 \rangle$

C_3 : $r_3(t) : [0,1] \longrightarrow \mathbb{R}^2 \quad \vec{r}_3(t) = \langle t, 2t \rangle$

(a) We start by computing the critical pts of f on $\mathbb{R}^2$:

$$f_x = \frac{-2x(2+2x^2y^2) - (2y^2-x^2)(4xy^2)}{(2+2x^2y^2)^2} = \frac{(-2x)(2+2x^2y^2+4y^4-2x^2y^2)}{(2+2x^2y^2)^2}$$

$$= \frac{-x(2y^4+1)}{(1+x^2y^2)^2}$$

$$f_y = \frac{4y(2+2x^2y^2) - (2y^2-x^2)(4yx^2)}{(2+2x^2y^2)^2} = \frac{4y(2+2x^2y^2-2x^2y^2+2x^4}{(2+2x^2y^2)^2}$$

$$= \frac{2y(1+x^4)}{(1+x^2y^2)^2}$$

So critical pts : $\nabla f_{(x,y)} = \vec{0} \iff x\underbrace{(2y^4+1)}_{>0} = 0 \ \& \ 2y\underbrace{(1+x^4)}_{>0} = 0$

$\iff \boxed{(x,y) = (0,0).}$

We use the Second Derivative Test to decide the nature of the only critical pt.

- $f_{xx} = \dfrac{(-2y^4-1)(1+x^2y^2)^2 + x(2y^4+1)\,2(1+x^2y^2)(2xy^2)}{(1+x^2y^2)^4} \rightsquigarrow f_{xx}(0,0) = \dfrac{-1}{1} = -1 < 0$

- $f_{xy} = f_{yx} = \dfrac{-8xy^3(1+x^2y^2)^2 + x(2y^4+1)(2(1+x^2y^2)(2x^2y))}{(1+x^2y^2)^4}$

  $\rightsquigarrow f_{xy}(0,0) = f_{yx}(0,0) = 0.$

- $f_{yy} = \dfrac{2(1+x^4)(1+x^2y^2)^2 - 2y(1+x^4)(2(1+x^2y^2)\,2x^2y)}{(1+x^2y^2)^4} \rightsquigarrow f_{yy}(0,0) = \dfrac{2}{1} = 2 > 0$

$\Rightarrow D_{(0,0)} = (f_{xx}f_{yy} - f_{xy}^2)_{(0,0)} = -1 \cdot 2 < 0 \quad [\ \& \ f_{xx}(0,0) = -1 < 0\ ]$

So $(0,0)$ is a saddle pt!

Notice: $f(x,0) = \frac{-x^2}{2} \leq 0$ $\quad f(0,0)=0 \Rightarrow (0,0)$ is a saddle pt in $\mathbb{R}^2$

$f(0,y) = \frac{2y^2}{2} = y^2 \geq 0$ (no need to use the test)

But these pts are not ∂ in R. Also (0,0) is in the boundary, so the test doesn't apply

(b) Boundary pts: We must analyze each one of the three curves that constitute the boundary of R.

(1) Curve C_1: $g_1(t) = f(\vec{r}_1(t)) = f(t,t) = \frac{2t^2-t^2}{2+2t^4} = \frac{t^2}{2(1+t^4)}$

$$g_1'(t) = \frac{2t\,2(1+t^4) - t^2\,8t^3}{(2(1+t^4))^2} = \frac{t(1+t^4-2t^4)}{(1+t^4)^2} = \frac{t(1-t^4)}{(1+t^4)^2}$$

- So $g_1'(t)=0 \iff t=0$ or $1=t^4 \leadsto t=\pm1$ but $0\leq t\leq 2$ so $t=0$ & $t=1$ are the ! critical pts; $g_1(1)=\boxed{\frac{1}{4}}$
- Extreme pts $g_1(0) = f(0,0) = \boxed{0}$ $\quad g_1(2)=f(2,2) = \frac{4}{2\cdot17} = \boxed{\frac{2}{17}}$

(2) Curve C_2: $g_2(t) = f(\vec{r}_2(t)) = f(t,2) = \frac{8-t^2}{2+8t^2} = \frac{8-t^2}{2(1+4t^2)}$

$$g_2'(t) = \frac{-2t\,2(1+4t^2)-(8-t^2)\,16t}{(2(1+4t^2))^2} = \frac{-t(1+4t^2)-4t(8-t^2)}{(1+4t^2)^2}$$

$$g_2'(t) = \frac{-t(1+4t^2+32-4t^2)}{(1+4t^2)^2} = \frac{-t\,33}{(1+4t^2)^2}$$

- So $g_2'(t)=0 \iff t=0$

But $1\leq t\leq 2$ so no critical pts!

- Extreme pts: $g_2(1) = f(1,2) = \frac{8-1}{2+8} = \boxed{\frac{9}{10}}$; $g_2(2) = f(2,2) = \boxed{\frac{2}{17}}$

(3) Curve C_3: $g_3(t) = f(\vec{r}_3(t)) = f(t,2t) = \frac{8t^2-t^2}{2+8t^4} = \frac{7t^2}{2(1+4t^4)}$

$$g_3'(t) = \frac{7}{2}\left(\frac{2t(1+4t^4)-t^2\,16t^3}{(1+4t^4)^2}\right) = \frac{7}{2}\,2t\,\frac{(1+4t^4-8t^4)}{(1+4t^4)^2} = \frac{7t(1-4t^4)}{(1+4t^4)^2}.$$

- So $g_3'(t)=0 \iff t=0$ or $1=4t^4 \iff t=0$ or $t=\pm\frac{1}{\sqrt{2}}$

But $0\leq t\leq 1 \leadsto$ only 2 solns $t=0$ & $t=\frac{1}{\sqrt{2}}$

$g_3(0) = f(0,0) = 0$, $\quad g_3\left(\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right) = \frac{7/2}{2\left(1+4\frac{1}{4}\right)} = \boxed{\frac{7}{8}}$

- Extreme pts: $g_3(0) = \boxed{0}$, $g_3(1) = f(1,2) = \boxed{\frac{9}{10}}$

Compare all to pick the winners:

$\boxed{\text{Absolute minimum} = f(0,0) = 0}$, $\boxed{\text{Absolute maximum} = f(1,2) = \frac{9}{10}}$

Conclude: (0,0) is NOT a saddle pt but a local minimum on R

(b) We use our previous analysis to detect what's the behavior of f on $\mathbb{R}^2$:

- We know $(0,0)$ is a saddle pt & there are no other critical pts., so no local extremal values.

$\boxed{\text{Since there are } \underline{\text{no}} \text{ local extrema \& } \mathbb{R}^2 \text{ is } \underline{\text{OPEN}} \Rightarrow \text{no absolute extremal values}}$

Solution 2:

- By our observation. $f(0,y) = \frac{2y^2}{2} = y^2 \xrightarrow[y \to +\infty]{} +\infty$ so there is no absolute maximum
- $f(x,0) = \frac{-x^2}{2} \xrightarrow[x \to +\infty]{} -\infty$ so there is no absolute minimum

Problem 4 The function is differentiable up to any order. We start by computing its critical points:

$$\nabla f(x,y) = \langle 2xy, x^2 \rangle = \vec{0} \iff x = 0$$

so we have a line of critical pts!

Notice $f(0,y) = -3$ for all y.

$$f_{xx} = 2y, \quad f_{yy} = 0, \quad f_{xy} = f_{yx} = 2x$$

$$D_{(x,y)} = f_{xx} f_{yy} - f_{xy}^2 = -(2x)^4 \rightsquigarrow D_{(0,y)} = 0$$

$\Rightarrow$ the Second Derivative Test is inconclusive

- We check nearby points to decide:

(1) On $\mathbb{R}^2$: $\begin{cases} f(x,y) = x^2y - 3 > 0 - 3 = -3 & \text{if } x, y > 0 \\ f(x,y) = x^2y - 3 < 0 - 3 & \text{if } x > 0 \text{ } \underline{\text{but}} \text{ } y < 0 \end{cases}$

so all pts in the line $x=0$ are saddle points! on $\mathbb{R}^2$

- To check extremal values, we check the behavior at infinity

Example: $x = y > 0$. $\lim_{x \to \infty} f(x,x) = \lim_{x \to +\infty} x^3 - 3 = +\infty \Rightarrow$ No absolute max value

$x = y < 0$ $\lim_{x \to -\infty} f(x,x) = \lim_{x \to -\infty} x^3 - 3 = -\infty \Rightarrow$ No absolute min value

$\boxed{\text{On } \mathbb{R}^2 \text{ there are no local or absolute extremal points, only a line } (x=0) \text{ of saddle points.}}$

(2) The situation on $R = \{(x,y) : y \geq 0\}$ is different since $x^2 y \geq 0$ on R

so $f(x,y) = x^2y - 3 \geq -3$ and equality occurs iff $x=0$ or $y=0$.

$\boxed{\text{So the lines } (x=0) \text{ \& } (y=0) \text{ are local and absolute minimum.}}$

By taking the limit along $x=y>0$ when $x \to +\infty$ we see f has no local nor abs. maximum

Problem 5 : We can solve this in 2 ways:

Soln 1 The identity constraint $xyz=4$ allows us to write z explicitly as a function of x, y provided $x, y \neq 0$. $z = z(x,y) = \frac{4}{xy}$ defined on $D = \mathbb{R}^2 \setminus \{x\text{-axis}\} \cup \{y\text{-axis}\}$.

Then $g(x,y) = f(x, y, \frac{4}{xy}) = x^2 + y^2 + \left(\frac{4}{xy}\right)^2 = \frac{x^4y^2 + x^2y^4 + 16}{x^2y^2}$

and g is defined on D. Also g is differentiable up to any order on D

- We compute the critical values of g:

$$\begin{cases} g_x = 2x + \frac{16}{y^2} \cdot \frac{(-2)}{x^3} = \frac{2(y^2x^4 - 16)}{x^3y^2} \\ g_y = 2y + \frac{16}{x^2} \frac{(-2)}{y^3} = \frac{2(x^2y^4 - 16)}{x^3y^2} \end{cases}$$

$\nabla g(x,y) = \vec{0} \iff y^2x^4 = 16$ & $x^2y^4 = 16$

$(yx^2-4)(yx^2+4) = 0$ & $(xy^2-4)(xy^2+4) = 0$.

We have 4 situations to check:

① $\begin{cases} yx^2 - 4 = 0 \\ xy^2 - 4 = 0 \end{cases} \Rightarrow xy^2 = xy^2 \Rightarrow xy(y-x) = 0 \underset{x,y\neq 0}{\Longrightarrow} \boxed{x=y}$ & $x^3 = 4$ $\boxed{x = \sqrt[3]{4}}$

② $\begin{cases} yx^2 - 4 = 0 \\ y^2x + 4 = 0 \end{cases} \Rightarrow yx^2 = -y^2x \Rightarrow xy(x+y) = 0 \Longrightarrow \boxed{x=-y}$ & $-x^3 = 4$ $\boxed{x = -\sqrt[3]{4}}$

③ $\begin{cases} yx^2 + 4 = 0 \\ xy^2 - 4 = 0 \end{cases} \Rightarrow yx^2 = -y^2x \overset{\text{from ②}}{\Longrightarrow} \boxed{x=-y}$ & $x^3 = 4$ $\boxed{x = \sqrt[3]{4}}$

④ $\begin{cases} yx^2 + 4 = 0 \\ xy^2 + 4 = 0 \end{cases} \Rightarrow yx^2 = xy^2 \overset{\text{from ①}}{\Longrightarrow} \boxed{x=y}$ & $x^3 = 4$ $x = -\sqrt[3]{4}$

So we have $\boxed{4 \text{ critical points}: (\sqrt[3]{4}, \sqrt[3]{4}), (\sqrt[3]{4}, -\sqrt[3]{4}), (-\sqrt[3]{4}, \sqrt[3]{4}), (-\sqrt[3]{4}, -\sqrt[3]{4})}$.

& they all give the same value for $g(x,y) = 2\sqrt[3]{16} + \frac{16}{(\sqrt[3]{4})^4} = 2\sqrt[3]{16} + \frac{16}{\sqrt[3]{8^2 2^2}} = 2\sqrt[3]{16} + \frac{16}{4\sqrt[3]{4}}$

$= 2 \cdot 2\sqrt[3]{2} + (\sqrt[3]{4})^2 = 4\sqrt[3]{2} + 2\sqrt[3]{2} = \boxed{6\sqrt[3]{2}}$

- We use the Second Derivative Test:

$$g_{xx} = 2 + \frac{96}{y^2x^4}$$
$$g_{xy} = g_{yx} = \frac{64}{x^3y^3}$$
$$g_{yy} = 2 + \frac{96}{x^2y^4}$$

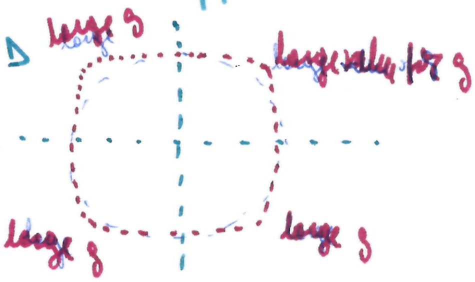
$$D = g_{xx}g_{yy} - g_{xy}^2 = \left(2+\frac{96}{y^2x^4}\right)\left(2+\frac{96}{y^4x^2}\right) - \frac{64^2}{x^6y^6}$$
$$= \left(4 + \frac{96^2}{x^6y^6} + \frac{2\cdot 96}{x^2y^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\right) - \frac{64^2}{x^6y^6}$$
$$= \frac{1}{x^2y^2}\left(4 + \frac{(96^2-64^2)}{x^4y^4} + \frac{2\cdot 96\,(x^2+y^2)}{x^2y^2}\right) \cdots$$

$$D = \frac{1}{x^2y^2}\left(4 + \frac{5120}{x^4y^4} + \frac{192\,(x^2+y^2)}{x^2y^2}\right) = \frac{4}{x^2y^2}\left(\frac{x^4y^4+1280}{x^4y^4} + \frac{48\,(x^2+y^2)}{x^2y^2}\right)$$

For **all** $x, y \neq 0$, we see that $D_{(x,y)} > 0$ (it is a sum of non-negative terms & $1280 > 0$)
$g_{xx} > 0$ also (same reason)

From this we conclude that all 4 critical pts. are local minima.

To decide the existence or not of absolute extreme values we must study the limits as we approach the boundary of D, as well as approaching ~~infinity~~



- $g(x,y) = \underbrace{(x^2+y^2)}_{\geq 0} + \boxed{\frac{16}{x^2y^2}}$ and $\frac{16}{x^2y^2} \longrightarrow +\infty$ whenever $x \to 0$ or $y \to 0$

So $g(x,y)$ has no absolute maximum

- On the other hand: $\frac{16}{x^2y^2} \longrightarrow 0$ whenever $x \to +\infty$ for y fixed.
- So if $x, y \to +\infty$ along any path $\lim_{x,y\to+\infty} g(x,y) = +\infty$

Since when going to the boundary: $g(x,y) \longrightarrow +\infty$, we conclude that g has an absolute minima. Since D is open, the abs minima are also local minima.

We conclude g has no absolute maxima but it has abs minima

The calculation thus shows that g has no absolute maximum value but it has 4 local minima: $(\pm\sqrt[3]{4}, \sqrt[3]{4})$ & $(\pm\sqrt[3]{4}, -\sqrt[3]{4})$, which are also absolute minima.

- Notice that the surface $xyz=4$ is unbounded and close, so a priori there was no reason to expect extremal values.

Soln 2: We use Lagrange multipliers (f & g are differentiable up to arbitrary order)

$$\begin{cases} f(x,y) = x^2+y^2+z^2 \rightsquigarrow \nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle \\ g(x,y,z) = xyz-4 \rightsquigarrow \nabla g(x,y,z) = \langle yz, xz, xy\rangle \neq \vec{0} \end{cases}$$

if $xyz=4$. (because $x,y,z\neq 0$)

We need to find $(x,y,z;\lambda)$ such that:

$$(*)\begin{cases}\begin{cases} 2x = \lambda yz & (1)\\ 2y = \lambda xz & (2)\\ 2z = \lambda xy. & (3)\end{cases}\\ xyz=4\end{cases}$$

$\Rightarrow$ multiply each eqn by x,y,z resp and use $xyz=4$.

$$\begin{cases} 2x^2 = \lambda 4\\ 2y^2 = \lambda 4\\ 2z^2 = \lambda 4\\ xyz=4\end{cases} \Rightarrow 2\lambda = x^2 = y^2 = z^2$$

$(x^2)^3 = (xyz)^2 = 16$

$\Rightarrow \boxed{x = \pm\sqrt[3]{4}}$ (same for y, z).

& $\boxed{\lambda = +\sqrt[3]{2}}$

We have up to 8 candidates. We go back to the original system $(*)$ to check if they are all solutions or if some of them should be discarded.

- $x = \pm\sqrt[3]{4}$, $y = \sqrt[3]{4}$, $z = \sqrt[3]{4}$ $\rightsquigarrow$ $xyz = -4$ unless $x = \sqrt[3]{4}$
- $x = \pm\sqrt[3]{4}$, $y = -\sqrt[3]{4}$, $z = -\sqrt[3]{4}$ $\rightsquigarrow$ $xyz = -4$ unless $x = \sqrt[3]{4}$.
- $x = \pm\sqrt[3]{4}$, $y = \sqrt[3]{4}$, $z = -\sqrt[3]{4}$ $\rightsquigarrow$ $xyz = -4$ " $x = -\sqrt[3]{4}$
- $x = \pm\sqrt[3]{4}$, $y = -\sqrt[3]{4}$, $z = \sqrt[3]{4}$ $\rightsquigarrow$ $xyz = -4$ " $x = -\sqrt[3]{4}$

So 4 candidates:

1. $x = \sqrt[3]{4} = y = z$ $\rightsquigarrow$ $2x = 2\sqrt[3]{4} \overset{?}{=} \sqrt[3]{2}\cdot\sqrt[3]{4}\sqrt[3]{4} = \sqrt[3]{32} = 2\sqrt[3]{4}$ ✓
   Other 2 eqns are the same ✓
2. $x = \sqrt[3]{4}$, $y = z = -\sqrt[3]{4}$ $\rightsquigarrow$ $2x = 2\sqrt[3]{4} \overset{?}{=} \sqrt[3]{2}(-\sqrt[3]{4})(-\sqrt[3]{4}) = 2\sqrt[3]{4}$ ✓
   $2y = -2\sqrt[3]{4} \overset{?}{=} \sqrt[3]{2}(\sqrt[3]{4})(-\sqrt[3]{4}) = -2\sqrt[3]{4}$ ✓ same for eq (3)
3. $x = -\sqrt[3]{4} = z$, $y = \sqrt[3]{4}$ $\rightsquigarrow$ same as (2)
4. $x = -\sqrt[3]{4} = y$, $z = \sqrt[3]{4}$ $\rightsquigarrow$ " " (3)

So 4 candidates & for all we get the same value for $f = 3(\sqrt[3]{4})^2 = \boxed{6\sqrt[3]{2}}$

We don't know if these points are local max/min or saddle points.

Notice that $f(x,y,z) \geq 0$ so it's bounded below. There's a hope to have an absolute minima. But f can be as large as we want.

Eg $x = y$ $\quad z = \frac{4}{x^2}$ $\rightsquigarrow$ $f(x, x, \frac{4}{x^2}) = 2x^2 + \frac{16}{x^4} \xrightarrow[x \to +\infty]{} +\infty$ ✓

Since $xyz = 4$ has no boundary, an if any of x, y or $z \to +\infty$ we conclude $f(x,y,z) \longrightarrow +\infty$ there is no extremal max value but there is an absolute minima, which will be local minima as well. We conclude the 4 pts obtained are local & absolute minima. (Notice: These are the same we computed with our first method).

Problem 6: f is differentiable up to any order, so we start by finding the critical pts of f:

$$\nabla f = \langle 4x^3 + 8xy - 16x, \; 4x^2 + 16y - 16 \rangle = \langle x(4x^2 + 8(y-2)), \; 4x^2 + 16(y-1) \rangle$$

So $\nabla f \neq \vec{0} \iff x(4x^2 + 8(y-2)) = 0 \;\; \& \;\; 4x^2 + 16(y-1) = 0$

- $x = 0 \implies 16(y-1) = 0$ so $y = 1$
- $x \neq 0$ then $\begin{cases} 4x^2 + 8(y-2) = 0 \\ 4x^2 + 16(y-1) = 0 \end{cases} \implies 8(y-2) = 16(y-1)$
  $y - 2 = 2(y-1) \implies \boxed{y = 0}$

  & $4x^2 + 8(-2) = 4x^2 - 16 = 0 \implies x^2 = 4 \implies x = \pm 2$

3 critical points: $(0,1)$, $(\pm 2, 0)$.

We use the Second Derivative Test:

$$\begin{cases} f_{xx} = 12x^2 + 8y - 16 \\ f_{xy} = f_{yx} = 8x \\ f_{yy} = 16 \end{cases} \implies D = 16(12x^2 + 8y - 16) - (8x)^2$$

$$= (16 \cdot 12 - 64)x^2 + 16(8y - 16)$$

$$= 16(8x^2 + 8y - 16)$$

$D(0,1) = -128 < 0 \implies (0,1)$ is a saddle point!

$D(2,0) = 16^2 > 0 \quad f_{xx}(2,0) = 32 > 0 \implies (2,0)$ is a local minimum

$D(-2,0) = 16^2 > 0 \quad f_{xx}(-2,0) = 32 > 0 \implies (-2,0)$ " " " "

To find the max / min on the boundary, we use Lagrange multipliers:

$g(x,y) = x^2+y^2-4$ is differentiable up to any order. Also, the boundary is closed & bounded, so there are extremal values.

$\nabla g = \langle 2x, 2y\rangle \neq \vec{0}$ on the boundary ✓

By Lagrange mult, the extremal values (x,y) & λ satisfy:

$$\begin{cases} x(4x^2+8(y-2)) = 2\lambda x & \rightarrow x=0 \text{ or } 4x^2+8(y-2)=2\lambda \\ 4x^2+16(y-1) = 2\lambda y \\ x^2+y^2=4 \end{cases}$$

• If $x=0$, then $\begin{cases} 16(y-1) = 2\lambda y \\ y^2=4 \rightarrow y=\pm 2 \end{cases} \Longrightarrow$ • $y=2$: $16 = 4\lambda$ $\boxed{\lambda=4}$ • $y=-2$ $-48=-4\lambda$ $\boxed{\lambda=12}$

• If $x\neq 0$, then $\begin{cases} 4x^2+8(y-2)=2\lambda & (1) \\ 4x^2+16(y-1)=2\lambda y & (2) \\ x^2+y^2=4 \end{cases}$ $\rightarrow x^2=4-y^2$ & substitute in (1) & (2)

$$\begin{cases} 4(4-y^2)+8(y-2)=2\lambda & (3) \\ 4(4-y^2)+16(y-1)=2\lambda y \end{cases}$$

$2\lambda - 8(y-2) = 2\lambda y - 16(y-1)$

$\boxed{2\lambda(1-y) = -8y}$

$\boxed{\lambda = \frac{-8y}{1-y}}$ and $y\neq 1$

Plug in (3) to solve for y:

$$4(4-y^2)+8(y-2) = \frac{-8y}{1-y}$$

$$4\left((4-y^2)(1-y)+8(y-2)(1-y)\right)+8y=0$$

Divide by 4:

$$(4-4y-y^2+y^3)+2(-2-3y+y^2)+2y=0$$

$$y(-8-3y+y^2)=0$$

$$0 = y\left(\left(y-\tfrac{3}{2}\right)^2-8-\tfrac{9}{4}\right) = y\left(y-\frac{3+\sqrt{41}}{2}\right)\left(y-\frac{3-\sqrt{41}}{3}\right)$$

Since $x^2+y^2=4$ we know $|y|<2$

• $\frac{3+\sqrt{41}}{2} > 2$ because $\sqrt{41} > 4-3=1$ ✓ so that root doesn't count

• $-2 < \frac{3-\sqrt{41}}{2}$ because $-7<-\sqrt{41} \iff \sqrt{41}<\sqrt{49}$ ✓ $\frac{3-\sqrt{41}}{2}<0$ because $3=\sqrt{9}<\sqrt{41}$

- So $y=0$ $x=\pm 2$ is a solution (We know it from critical pts calculation)
- $y = \frac{3-\sqrt{41}}{2} \Rightarrow x^2 = 4-y^2 = 4-\frac{(9+41-6\sqrt{41})}{4} = \frac{-34+6\sqrt{41}}{4} = \frac{-17+3\sqrt{41}}{2}$

  $x = \pm \frac{\sqrt{-34+6\sqrt{41}}}{2}$
- We compare the values of f at all 2 points to find the extremal values:

$f(0,1) = 8-16+1 = -7$

$f(\pm 2,0) = -15$

$$f\left(\pm\frac{\sqrt{-34+6\sqrt{41}}}{2}, \frac{3-\sqrt{41}}{2}\right) = \frac{(-34+6\sqrt{41})^2}{16} + 4\,\frac{3-\sqrt{41}}{8}(-34+6\sqrt{41}) - \frac{8(-34+6\sqrt{41})}{4} + 8\,\frac{(3-\sqrt{41})^2}{4} - 16(3-\sqrt{41}) + 1$$

$$= \frac{34^2 + 36\cdot 41 - 408\sqrt{41} + 8(\overbrace{-102-6\cdot 41}^{=348} + 52\sqrt{41}) + 1088 - 192\sqrt{41} + 32(9+41-6\sqrt{41}) - 768 + 256\sqrt{41} + 16}{16}$$

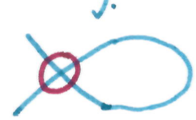
$$= \frac{1784-120\sqrt{41}}{16} \simeq 63.47$$

$\Rightarrow$ Absolute maximum at $\left(\pm\frac{\sqrt{-34+6\sqrt{41}}}{2}, \frac{3-\sqrt{41}}{2}\right)$

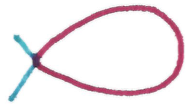
Absolute minimum at $(\pm 2, 0)$

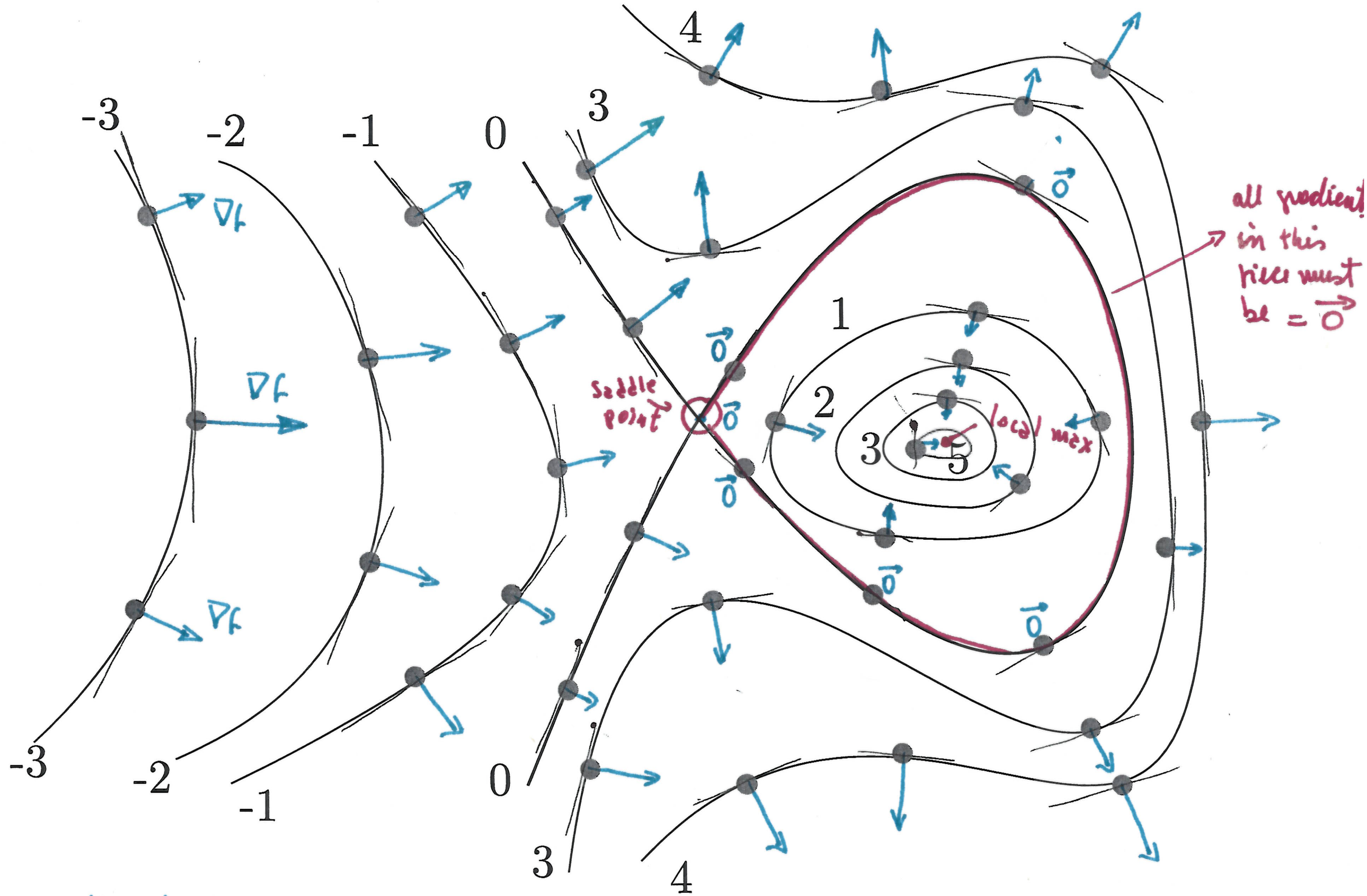
Problem 7: In order to draw the gradients, we ~~realize~~ remember that $\nabla f_{(a,b)}$ is normal to the tangent lines at the level curve $f_{(x,y)} = f_{(a,b)}$ at the point (a,b). & points in the direction in which f is increasing.

Also, if a level curve has a self crossing, then it has no tangent line at the self crossing, so f must have a saddle point there and $\nabla f_{(a,b)} = \vec{0}$ (it's a critical point & f is differentiable).

From the picture we detect a single potential saddle point: 

The level curves constructed suggest this point is indeed a saddle point (not a min & not a max because the level curves to the left are lower & to the right are higher, respectively.

• since all pts in  grow in both directions perpendicular to the tangent lines, there is not a unique increasing direction and we conclude $\nabla \vec{f} = 0$ on all these points.



• Inside the level curve $z = 5$ there must be a local maximum.