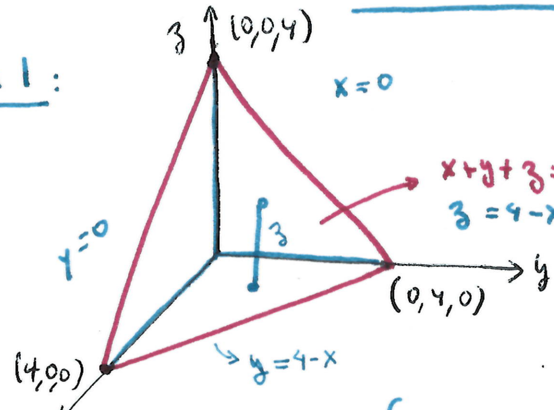
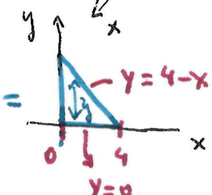
# Recitation IX (03/24/16)

## Problem 1:




• 2 densities:
$$\begin{cases} \rho_1(x,y,z) = 8 - z \\ \rho_2(x,y,z) = 4 + z \end{cases}$$

$z = 0 = G(x,y)$ We describe the solid as

$R =$ 


region $R$

$D = \{(x,y,z) : 0 \le z \le 4-x-y, \boxed{0 \le y \le 4-x, \ 0 \le x \le 4}\}$

(1) We compute the masses: $m_1 = \iiint_D 8-z \, dV = \int_0^4 \left( \int_0^{4-x} \left( \int_0^{4-x-y} 8-z \, dz \right) dy \right) dx$

$$= \underbrace{8 \int_0^4 \int_0^{4-x} \int_0^{4-x-y} 1 \, dz \, dy \, dx}_{Ⓐ} - \underbrace{\int_0^4 \int_0^{4-x} \int_0^{4-x-y} z \, dz \, dy \, dx}_{Ⓑ}$$

$$Ⓐ = \int_0^4 \int_0^{4-x} (4-x-y) \, dy \, dx = \int_0^4 (4-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=4-x} dx = \int_0^4 (4-x)(4-x) - \frac{(4-x)^2}{2} \, dx$$

$$= \int_0^4 \frac{(4-x)^2}{2} \, dx = -\frac{(4-x)^3}{3} \Big|_{x=0}^{x=4} = \frac{4^3}{3} = \frac{64}{3} = \text{Vol (Tetrahedron)}$$

[see Recitation 8]

$$Ⓑ = \int_0^4 \int_0^{4-x} \frac{(4-x-y)^2}{2} \, dy \, dx = \frac{1}{2} \int_0^4 \int_0^{4-x} (16 - 2x + x^2 + y^2 - 2y(4-x)) \, dy \, dx$$

$$= \frac{1}{2} \int_0^4 (16 - 2x + x^2) y + \frac{y^3}{3} - y^2(4-x) \Big|_{y=0}^{y=4-x} dx$$

$$= \frac{1}{2} \int_0^4 (4-x)^3 + \frac{(4-x)^3}{3} - (4-x)^3 \, dx = \frac{-1}{6} \frac{(4-x)^4}{4} \Big|_{x=0}^{x=4} = \frac{4^3}{6} = \frac{32}{3}.$$

So $m_1 = 8 \cdot \frac{64}{3} - \frac{32}{3} = \boxed{\frac{480}{3}} = \boxed{160}$

• $m_2 = \iiint_D 4 + z \, dV = 4 Ⓐ + Ⓑ = 4 \cdot \frac{64}{3} + \frac{32}{3} = \boxed{\frac{288}{3}} = \boxed{96}$

(2) We next compute the center of mass of each solid

• Solid 1: $M_{1,xy} = \iiint\limits_D z(8-z)\,dV = 8\underbrace{\iiint\limits_D z\,dV}_{=B} - \underbrace{\iiint\limits_D z^2\,dV}_{=C}$

$$C = \iiint\limits_D z^2\,dV = \int_0^4\int_0^{4-x}\int_0^{4-x-y} z^2\,dz\,dy\,dx = \int_0^4\int_0^{4-x} \frac{z^3}{3}\Big|_{z=0}^{z=4-x-y} dy\,dx$$

$$= \int_0^4\int_0^{4-x} \frac{(4-x-y)^3}{3}\,dy\,dx = \frac{1}{3}\int_0^4 \frac{(4-x-y)^4}{-4}\Big|_{y=0}^{y=4-x} dx = \frac{1}{12}\int_0^4 (4-x)^4\,dx$$

$$= \frac{1}{12}\,\frac{(4-x)^5}{-5}\Big|_{x=0}^{x=4} = \frac{4^5}{12\cdot 5} = \frac{4^4}{5} = \frac{256}{5}$$

so $M_{1,xy} = 8\cdot\frac{32}{3} - \frac{256}{5} = \frac{512}{15} \rightsquigarrow \bar{z} = \frac{512/15}{160} = \boxed{\frac{16}{75}}$

$$M_{1,xz} = \iiint\limits_D y(8-z)\,dV = 8\iiint\limits_D y\,dV - \iiint\limits_D yz\,dV$$

• D is symmetric in $x, y, z$ so $\iiint\limits_D z\,dV = \iiint\limits_D y\,dV = \iiint\limits_D x\,dV = \frac{32}{3}$

$$\iiint\limits_D yz\,dV = \int_0^4\int_0^{4-x}\int_0^{4-x-y} yz\,dz\,dy\,dx = \int_0^4\int_0^{4-x} y\frac{z^2}{2}\Big|_{z=0}^{z=4-x-y} dy\,dx = \int_0^4\int_0^{4-x} y\frac{(4-x-y)^2}{2}\,dy\,dx$$

$$= \int_0^4\int_0^{4-x} \frac{y\left((4-x)^2+y^2-2y(4-x)\right)}{2}\,dy\,dx = \frac{1}{2}\int_0^4 \frac{y^2}{2}(4-x)^2 + \frac{y^4}{4} - \frac{2y^3}{3}(4-x)\Big|_{y=0}^{y=4-x} dx$$

$$= \frac{1}{2}\int_0^4 \frac{(4-x)^4}{2} + \frac{(4-x)^4}{4} - \frac{2}{3}(4-x)^4\,dx = \frac{5}{24}\int_0^4 (4-x)^4\,dx = -\frac{(4-x)^5}{24}\Big|_{x=0}^{x=4} = \frac{4^5}{24} = \frac{128}{3}$$

so $M_{1,xz} = 8\cdot\frac{32}{3} - \frac{128}{3} = \frac{128}{3} \rightsquigarrow \bar{y} = \frac{128/3}{160} = \boxed{\frac{4}{15}}$

$$M_{1,yz} = \iiint\limits_D x(8-z)\,dV = 8\iiint\limits_D x\,dV - \iiint\limits_D xz\,dV$$

By symmetry of D & $xz$ with respect to the change $x \leftrightarrow y$, we get $\iiint\limits_D xz\,dV = \iiint\limits_D yz\,dV$

So $M_{1,yz} = 8\,\frac{32}{3} - \frac{128}{3} = \frac{128}{3} \rightsquigarrow \bar{x} = \boxed{\frac{4}{15}}$

$$\boxed{\text{The center of mass of the solid}_1 \text{ is } \left(\tfrac{4}{15}, \tfrac{4}{15}, \tfrac{16}{75}\right).}$$

Solid 2 : $M_{2,xy} = \iiint_D z(4+z)\,dV = 4\iiint_D z\,dV + \iiint_D z^2\,dV$

use calculations for solid 1

$= 4\cdot\frac{32}{3} + \frac{256}{5} = \frac{1408}{15}$. $\rightsquigarrow \bar{z} = \frac{\frac{1408}{15}}{96} = \boxed{\frac{44}{45}}$

$M_{2,xz} = \iiint_D y(4+z)\,dV = 4\iiint_D y\,dV + \iiint_D yz\,dV$

$= 4\cdot\frac{32}{3} + \frac{128}{3} = \frac{256}{3}$ $\rightsquigarrow \bar{y} = \frac{\frac{256}{3}}{96} = \boxed{\frac{4}{9}}$

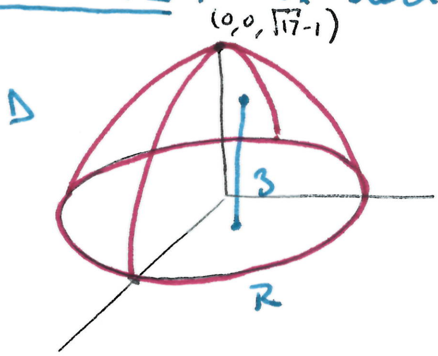
$M_{2,yz} = \iiint_D x(4+z)\,dV = 4\iiint_D x\,dV + \iiint_D xz\,dV = \frac{256}{3}$

use symmetry

$\rightsquigarrow \bar{x} = \boxed{4/9}$

**The center of mass of the solid 2 is $\left(\frac{4}{9}, \frac{4}{9}, \frac{44}{45}\right)$**

Problem 2 : We start by drawing the solid.




the graph meets the xy-plane where

$0 = \sqrt{17} - \sqrt{1+x^2+y^2}$, so $x^2+y^2 = 16 = 4^2$.

$R = \{x^2+y^2 \le 4^2\}$ → suitable for polar coordinates $R = \{(r,\theta)\ \ 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

The hyperboloid is suitable for cylindrical coordinates

$0 \le z \le \sqrt{17} - \sqrt{1+x^2+y^2} = \sqrt{17} - \sqrt{1+r^2}$.

$D = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 4,\ 0 \le z \le \sqrt{17} - \sqrt{1+r^2}\}$

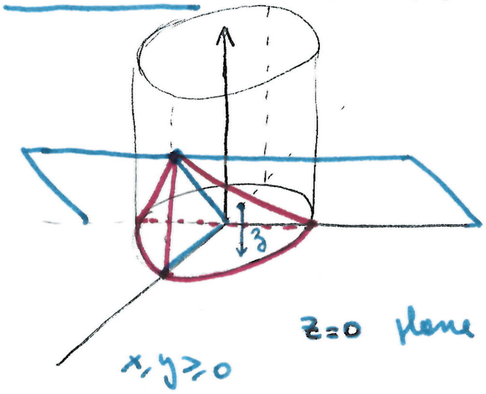
$$\mathrm{Vol}(D) = \int_0^{2\pi}\left(\int_0^4\left(\int_0^{\sqrt{17}-\sqrt{1+r^2}} 1\cdot r\,dz\right)dr\right)d\theta$$

$$= \int_0^{2\pi}\int_0^4 r\left(\sqrt{17} - \sqrt{1+r^2}\right)dr\,d\theta = 2\pi\int_0^4 \sqrt{17}\,r - r\sqrt{1+r^2}\,dr$$

$$= 2\pi\left(\sqrt{17}\frac{r^2}{2} - \frac{(1+r^2)^{3/2}}{\frac{3}{2}\cdot 2}\right)\Bigg|_{r=0}^{r=4} = 2\pi\left(\sqrt{17}\,8 - \frac{17^{3/2}}{3} - \left(0 - \frac{1}{3}\right)\right)$$

$$= 2\pi\left(\sqrt{17}\frac{7}{3} + \frac{1}{3}\right) = \boxed{\frac{2}{3}\pi\left(1 + 7\sqrt{17}\right)}$$

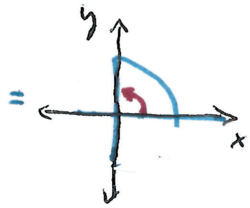
**Problem 3**: We start by drawing the solid.




We compute the curve of intersection between the plane $z=x$ & the cylinder $= \begin{cases} x^2+y^2=1 \\ z=x \end{cases}$ ⇝ lift the circle to height $x$

The $\text{bounds}$: $G(x,y) = 0 \le z \le x = H(x,y)$

The projection of D to the xy-plane is the quarter-circle R =

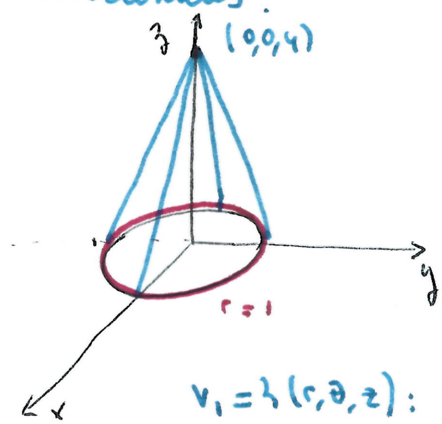


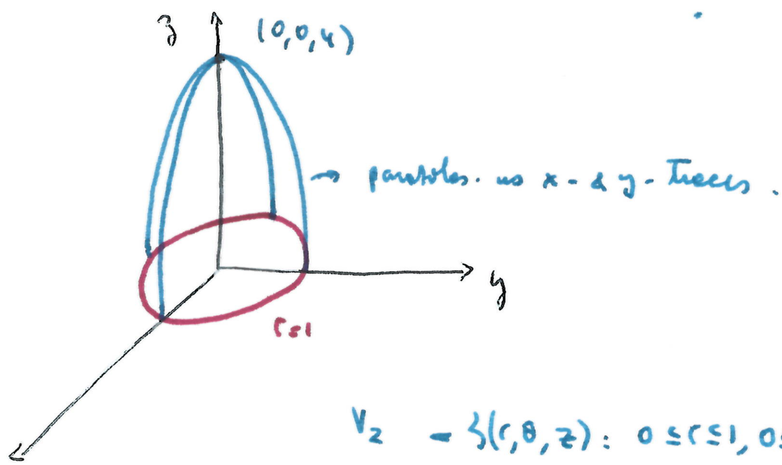

so it's a type II region. We can also write it in polar coordinates $R = \{(r,\theta) \quad 0 \le r \le 1, \quad \le \theta \le \frac{\pi}{2}\}$ ⇝ rectangular!

$$\mathrm{Vol}(D) = \int_0^{\pi/2}\int_0^1 \left(\int_0^{r\cos\theta} 1 \cdot r\, dz\right) dr \,)\, d\theta = \int_0^{\pi/2}\int_0^1 r^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left.\frac{r^3}{3}\right|_{r=0}^{r=1} dr\, d\theta = \int_0^{\pi/2} \frac{\cos\theta}{3}\, d\theta = \left.\frac{\sin\theta}{3}\right|_{\theta=0}^{\theta=\pi/2} = \boxed{\frac{1}{3}}.$$

**Problem 4**: In this case, the description of both solids is in cylindrical coordinates.




$V_1 = \{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4-4r\}$

$V_2 = \{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4-4r^2\}$

$$m_1 = \iiint_{V_1} (10-2z)\, dV = \int_0^1\int_0^{2\pi}\int_0^{4-4r} (10-2z)\, r\, dz\, d\theta\, dr = 2\pi \int_0^1\int_0^{4-4r} (10-2z)\, r\, dz\, dr$$

$$= 2\pi \int_0^1 \left. 10rz - rz^2 \right|_{z=0}^{z=4-4r} dr = 2\pi \int_0^1 10r(4-4r) - r(4-4r)^2\, dr$$

$$= 2\pi \int_0^1 40r - 40r^2 - 16r - 16r^3 + 32r^2 \, dr = 2\pi \int_0^1 24r - 8r^2 - 16r^3 \, dr$$

$$= 2\pi \left( 12r^2 - \frac{8}{3} r^3 - 4r^4 \Big|_{r=0}^{r=1} \right) = 2\pi \left( 12 - \frac{8}{3} - 4 \right) = \boxed{\frac{32\pi}{3}}$$

$$m_2 = \iiint_{D_2} (10-2z)\, dV = \int_0^1 \int_0^{2\pi} \int_0^{4-4r^2} (10-2z)\, r \, dz \, d\theta \, dr = 2\pi \int_0^1 \int_0^{4-4r^2} (10-2z)\, r \, dz \, dr$$

$$= 2\pi \int_0^1 10rz - rz^2 \Big|_{z=0}^{z=4-4r^2} dr = 2\pi \int_0^1 10r(4-4r^2) - r(4-4r^2)^2 \, dr$$

$$= 2\pi \int_0^1 40r - 40r^3 - 16r - 16r^5 + 32r^3 \, dr = 2\pi \int_0^1 40r - 8r^3 - 16r - 16r^5 \, dr$$

$$= 2\pi \left( 20r^2 - 2r^4 - 8r^2 - \frac{16}{6} r^6 \Big|_{r=0}^{r=1} \right) = 2\pi \left( 20 - 2 - 8 - \frac{8}{3} \right) = \boxed{\frac{44\pi}{3}}$$

The second solid has greater mass.

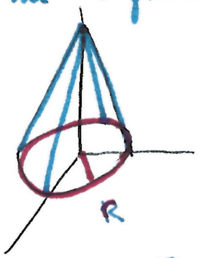
Observe: For all $0 \le r \le 1$, we have that $4 - 4r \le 4 - 4r^2$ (because $r^2 \le r$ if $0 \le r \le 1$). This means that the cone is contained in the paraboloid. Since the density function is positive on $D_2$, we conclude

$$\iiint_{D_2} \rho(x,y,z)\, dV = \iiint_{D_1} \rho(x,y,z)\, dV + \underbrace{\iiint_{D_2 \setminus D_1} \rho(x,y,z)\, dV}_{>0}$$

So the mass of the paraboloid is strictly greater than the mass of the cone.

**Problem 5**: Centroids account for objects with constant density. We use the symmetry of our solids to simplify the calculations.

(a)




The cone is symmetric about the $xz$-plane & $yz$-plane. So $\bar{x} = \bar{y} = 0$. In other words, the centroid lies in the positive $z$-axis. We assume $\rho \equiv 1$ to compute $\bar{z}$

$$M_{xy} = \iiint_D z \, dz = \int_0^R \left( \int_0^{2\pi} \left( \int_0^{R-r} z \, dz \right) d\theta \right) dr = 2\pi \int_0^R \frac{(R-r)^2}{2} \, dr$$

cone $= \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le R,\ 0 \le z \le R - r\}$

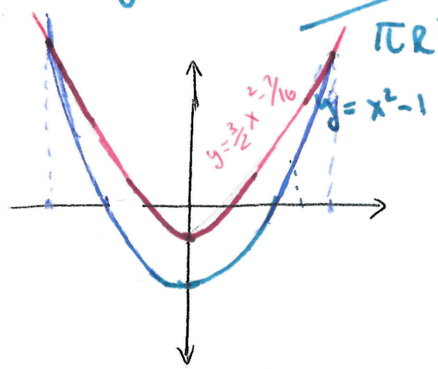
$$= \pi \int_0^R R^2 + r^2 - 2rR \, dr$$

$$= \pi \left(R^2 r + \frac{r^3}{3} - R r^2\right)\Bigg|_{r=0}^{r=R} = \pi\left(R^3 + \frac{R^3}{3} - R^3\right) = \boxed{\frac{\pi R^3}{3}}$$

$$\text{mass} = \iiint_D 1\, dz = \int_0^R \int_0^{2\pi}\int_0^{R-r} 1\, dz\, d\theta\, dr = 2\pi \int_0^R (R-r)\,dr = \frac{2\pi(R-r)^2}{-2}\Bigg|_{r=0}^{r=R} = \pi R^2$$

so $\bar z = \dfrac{\pi R^3/3}{\pi R^2} = \boxed{\dfrac{R}{3}}$ & $\boxed{\bar x = \bar y = 0}$.

(b)




The 2 curves meet when:

$$x^2 - 1 = \frac{3}{4}x^2 - \frac{7}{16} \iff \frac14 x^2 = +\frac{9}{16}$$

$$x^2 = \frac{9}{4} \Rightarrow x = \pm\frac32$$

so $y = x^2-1 = \frac94 - 1 = \frac54$.

The region is symmetric about the y-axis, so the centroid lies in the y-axis

$$\text{mass} = \iint_R 1\, dA = \int_{-3/2}^{3/2}\left(\int_{x^2-1}^{\frac34 x^2 - 7/16} dy\right)dx = \int_{-3/2}^{3/2} \left(\frac34 x^2 - \frac7{16}\right) - (x^2-1)\,dx$$

$$= \int_{-3/2}^{3/2} -\frac14 x^2 + \frac9{16}\,dx = -\frac{x^3}{12} + \frac{9}{16}x \Bigg|_{x=-3/2}^{x=3/2} = -\frac1{12}\left(\frac{27}{8} - \left(-\frac{27}{8}\right)\right) + \frac9{16}\, 2\cdot\frac32$$

$$= -\frac{27}{48} + \frac{27}{16} = \boxed{\frac98}$$

$$\bar y = \frac{M_x}{\text{mass}} \qquad M_x = \iint_R y\,dA = \int_{-3/2}^{3/2} \frac{y^2}{2}\Bigg|_{y=x^2-1}^{y=\frac34x^2-7/16} dx = \frac12\int_{-3/2}^{3/2} \left(\frac34x^2-\frac7{16}\right)^2 - (x^2-1)^2\,dx$$

$$= \frac12\int_{-3/2}^{3/2} \frac9{16}x^4 + \frac{49}{16^2} - \frac{21}{32}x^2 - x^4 - 1 + 2x^2\,dx = \frac12\int_{-3/2}^{3/2} -\frac{207}{16^2} + \frac{43}{32}x^2 - \frac{7}{16}x^4\,dx$$

$$= \frac12\left(-\frac{207}{16^2}x + \frac{43}{32\cdot3}x^3 - \frac{7}{16\cdot5}x^5\right)\Bigg|_{x=-3/2}^{x=3/2} = -\frac{207}{16^2}\frac32 + \frac{43}{32\cdot3}\frac{27}{8} - \frac{7}{16\cdot5}\frac{3^5}{2^5} = \boxed{\frac{-117}{320}}$$

Conclusion: $\bar y = \dfrac{-117/320}{9/8} = \boxed{-13/40}$ & $\boxed{\bar x = 0}$.