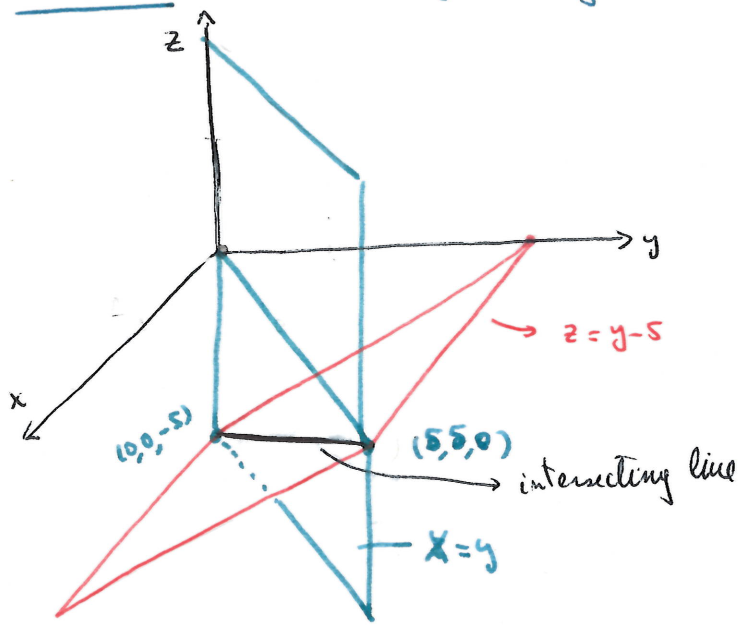
Recitation X

Problem 1 We start by drawing the picture of the region.

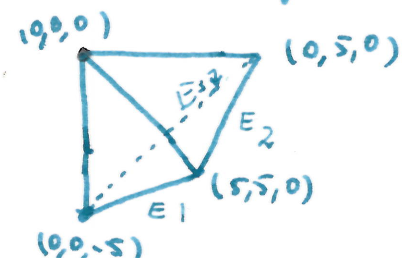


$z = y-5$ In the yz-plane it's a line!

$x = y$ In the xy-plane it's a line!

These 2 planes meet at the line through $(5,5,0)$ & $(0,0,-5)$

The solid D is a pyramide:



To integrate we need to check the sign of $x+y+z$. Easiest thing to do, slice D with the plane π $x+y+z=0$ & the 2 sides will then have different sign.

For this, we intersect this plane w/ the bounding planes of D, or even better, with the edges of D: Note that $(0,0,0)$ lies in the plane π, so we need only consider the edges on the plane $z = y-5$.

π ∩ $(z = y-5)$ gives $\begin{cases} x+y+z=0 \\ z = y-5 \end{cases}$, so $x+y+(y-5) = x+2y-5=0$

$\begin{cases} x = 5-2y \\ z = y-5 \end{cases}$ so we get the line through $(5,0,5)$ $(y=0)$ & $(-5,5,0)$ $(y=5)$

π ∩ (yz-plane) = gives $\begin{cases} y+z=0 \\ x=0 \end{cases}$ → line $y=-z$.

π ∩ ($x=y$ plane) = gives $\begin{cases} x+x+z=0 \\ x=y \end{cases}$ → $z=-2x$. line.

π ∩ ($z=0$ plane) = gives $\begin{cases} x+y=0 \\ z=0 \end{cases}$ → $x=-y$ line.

- $E_1 = (y-5=z \text{ plane}) \cap (x=y \text{ plane}) \Rightarrow \begin{cases} z=y-5 \\ z=-2x \\ x=5-2y \end{cases}$ is $E_1 \cap$ plane π

We get $z = y-5$
$z = -2(5-2y) = -10+4y$
so $y-5 = -10+4y$
$5 = 3y$ → $\boxed{y = 5/3}$

The point is $x = 5-2y = 5-2\frac{5}{3} = \frac{5}{3}$
$y = 5/3$
$z = 5/3 - 5 = -10/3$

$P_1 = \left(\frac{5}{3}, \frac{5}{3}, -\frac{10}{3}\right)$

• $E_2 = (y-5=z \text{ plane}) \cap (z=0 \text{ plane})$

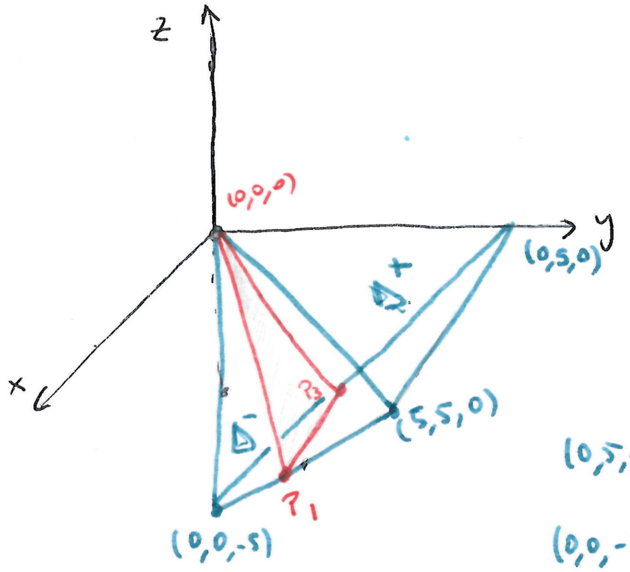
so $E_2 \cap$ plane π is $\begin{cases} z = y-5 \\ z=0 \\ x+y+z=0 \end{cases} \Rightarrow y=5, \quad x+5+0=0$, so $x=-5$

$P_2 = (-5, 5, 0)$ not in the solid!

• E_3 : $(y-5=z \text{ plane}) \cap (x=0)$

So $E_3 \cap$ plane π is $\begin{cases} z = y-5 \\ x=0 \\ x+y+z=0 \end{cases}$ so $\begin{cases} x=0 \\ y=z+5 \\ y=-z \end{cases}$ $\quad 2z=-5 \to z=-5/2$

$P_3 = (0, \frac{5}{2}, -\frac{5}{2})$



We get 3 points $(0,0,0)$, P_1 & P_3 to determine the plane π. This divides D in 2 pieces. (call them D^+ & D^-)

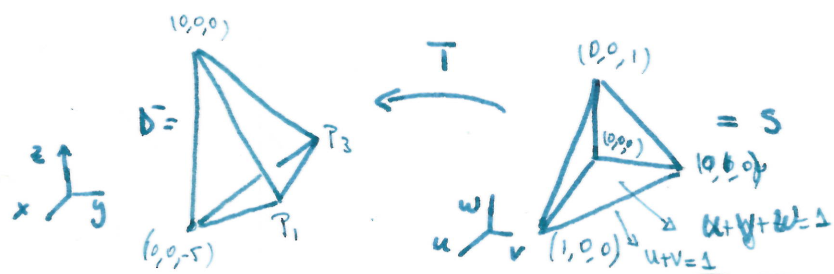
We determine the sign of $x+y+z$ in these pieces just by testing at the vertices.

$(0,5,0)$: $(x+y+z)_{(0,5,0)} = 5 > 0 \rightsquigarrow$ region to the right is D^+.

$(0,0,-5)$: $(x+y+z)_{(0,0,-5)} = -5 < 0 \rightsquigarrow$ region is D^-

$\text{Mass}(D) = \text{Mass}(D^+) + \text{Mass}(D^-)$

$$\text{Mass}(D^-) = \iiint_{D^-} -(x+y+z)+1 \; dV$$



We can calculate this integral in an easier way if we use a transformation T that sends the unit tetrahedron S to D^- by mapping $\begin{cases} (0,1,0) \longmapsto P_3 \\ (1,0,0) \longmapsto (0,0,5) \\ (0,0,0) \longmapsto (0,0,0) \\ (0,0,1) \longmapsto P_1 \end{cases}$

we write $T_{(u,v,w)} = \langle a_1 u + b_1 v + c_1 w + d_1, \; a_2 u + b_2 v + c_2 w + d_2, \; a_3 u + b_3 v + c_3 w + d_3 \rangle$

and try to find the constants (a_i, b_i, c_i, d_i) $i = 1,2,3$

• $T_{(0,0,0)} = (d_1, d_2, d_3) = (0,0,0)$ so all d_i's $=0$.

• $T(0,1,0) = (b_1, b_2, b_3) = P_3 = (0, 5/2, -5/2)$ so. $b_1 = 0$, $b_2 = 5/2$, $b_3 = -5/2$

• $T(0,0,1) = (c_1, c_2, c_3) = P_1 = (5/3, 5/3, -\frac{10}{3})$ so $c_1 = 5/3$, $c_2 = 5/3$, $c_3 = -\frac{10}{3}$

• $T(1,0,0) = (a_1, a_2, a_3) = (0,0,-5) \Rightarrow a_1 = 0$, $a_2 = 0$, $a_3 = -5$

Conclusion $T_{(u,v,w)} = \langle \frac{5}{3} w, \; \frac{5}{2} v + \frac{5}{3} w, \; -5u + (-\frac{5}{2}) v - \frac{10}{3} w \rangle$

$$J_{(u,v,w)} = \begin{vmatrix} 0 & 0 & 5/3 \\ 0 & 5/2 & 5/3 \\ -5 & -5/2 & -10/3 \end{vmatrix} = -\frac{5}{3}\left(0 - \left(\frac{5}{2} \cdot (-5)\right)\right) = -\frac{5^3}{6} = \boxed{-\frac{125}{6}}$$

Then: $\text{Mass}(D^-) = \iiint\limits_{D^-} -x-y-z+1\,dV = \iiint\limits_{S} \left(-\frac{5}{3}w - \frac{5}{2}v - \frac{5}{3}w + 5u + \frac{5}{2}v + \frac{10}{3}w + 1\right)\cdot\frac{125}{6}\,dV$

$$\begin{cases} x = \frac{5}{3}w \\ y = \frac{5}{2}v + \frac{5}{3}w \\ z = -5u - \frac{5}{2}v - \frac{10}{3}w \end{cases}$$

$$= \frac{125}{6}\iiint\limits_S (5u+1)\,dV = \frac{125}{6}\int_0^1\int_0^{1-u}\int_0^{1-u-v}(5u+1)\,dw\,dv\,du = \frac{125}{6}\int_0^1\int_0^{1-u}(5u+1)(1-u-v)\,dv\,du$$

$$= \frac{125}{6}\int_0^1\int_0^{1-u} 4u - 5uv - 5u^2 - v + 1\,dv\,du = \frac{125}{6}\int_0^1 4uv - \frac{5uv^2}{2} - 5u^2v - \frac{v^2}{2} + v\,\Big|_0^{1-u}\,du$$

$$= \frac{125}{6}\int_0^1 4u(1-u) - 5u\frac{(1-u)^2}{2} - 5u^2(1-u) - \frac{(1-u)^2}{2} + (1-u)\,du$$

$$= \frac{125}{6}\int_0^1 4u - 4u^2 - \frac{5u+5u^3-10u^2}{2} - 5u^2 + 5u^3 - \frac{1+u^2-2u}{2} + 1 - u\,du$$

$$= \frac{125}{6}\int_0^1 \frac{1}{2} + \frac{3}{2}u - \frac{9}{2}u^2 + \frac{5}{2}u^3\,du = \frac{125}{6}\left(\frac{1}{2}u + \frac{3u^2}{4} - \frac{3u^3}{2} + \frac{5}{8}u^4\,\Big|_0^1\right)$$

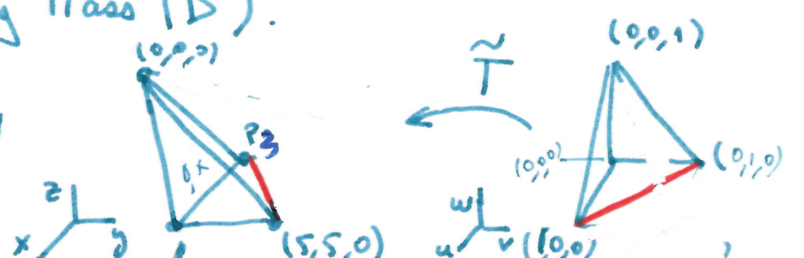
$$= \frac{125}{6}\cdot\frac{3}{8} = \boxed{\frac{125\cdot 3}{48}}$$

(*) Note $\frac{3}{8} = \iiint\limits_S 1\,dV + 5\iiint\limits_S u\,dV = \boxed{\frac{1}{6}} + 5\iiint\limits_S u\,dV$

Vol = unit tetrahedron

Similar calculation can be done for computing $\text{Mass}(D^+)$.

$$\text{Mass}(D^+) = \iiint\limits_{D^+} (x+y+z)+1\,dV$$



Now we need to break the square pyramid into 2 pieces that look like tetrahedra. D_1^+ & D_2^+

$\underline{D_1^+}$: We want $\tilde{T}$: $\tilde{T}(0,0,0) = (0,0,0)$ → same d's as before

$\tilde{T}(1,0,0) = (5,5,0)$ → $a_1 = 5$, $a_2 = 5$, $a_3 = 0$

$\tilde{T}(0,1,0) = P_3$ → same b's as before

$\tilde{T}(0,0,1) = P_1$ → " c's " "

$$\tilde{T}(u,v,w) = \left\langle 5u + \frac{5}{3}w,\ 5u + \frac{5}{2}v + \frac{5}{3}w,\ -\frac{5}{2}v - \frac{10}{3}w \right\rangle$$

$\Rightarrow x+y+z = 10u$

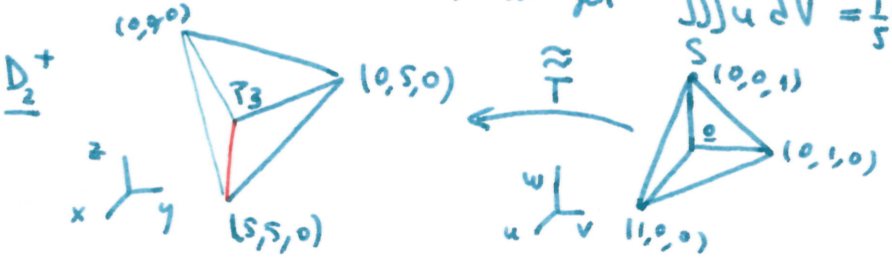
$$\tilde{J}(u,v,w) = \begin{vmatrix} 5 & 0 & 5/3 \\ 5 & 5/2 & 5/3 \\ 0 & -5/2 & -\frac{10}{3} \end{vmatrix} = 5\cdot\left(\frac{5}{2}\left(-\frac{10}{3}\right) - \frac{5}{3}\left(-\frac{5}{2}\right)\right) + \frac{5}{3}\left(5\cdot\left(-\frac{5}{2}\right)\right)$$

$$= -\frac{150\cdot 5}{6} = -125.$$

So $\text{mass}(D_1^+) = \iiint_S (10u+1)\,(125)\,dV = 125 \iiint_S (10u+1)\,dV$

$= 125\left(10\iiint_S u\,dV\right) + 125\iiint_S 1\,dV\,) = 125\left(\frac{10}{24} + \frac{1}{6}\right) = \frac{125\cdot 14}{24} = \frac{875}{12}$

From the note (*) we get $\iiint_S u\,dV = \frac{1}{5}\left(\frac{3}{8} - \frac{1}{6}\right) = \frac{1}{5}\,\frac{5}{24} = \boxed{\frac{1}{24}}$



Want $\tilde{\tilde{T}}$: $(0,0,0) \longmapsto (0,0,0)$
$(1,0,0) \longmapsto (5,5,0)$
$(0,1,0) \longmapsto P_3$
$(0,0,1) \longmapsto (0,0,5)$
$\boxed{c_1 = c_2 = 0,\ c_3 = 5}$

So $\tilde{\tilde{T}}(u,v,w) = \langle 5u,\ 5u + \frac{5}{2}v,\ -\frac{5}{2}v + 5w \rangle$

$\tilde{\tilde{J}}(u,v,w) = \begin{vmatrix} 5 & 0 & 0 \\ 5 & 5/2 & 0 \\ 0 & -5/2 & 5 \end{vmatrix} = 5\cdot 5\cdot \frac{5}{2} = \frac{125}{2}.$ $\qquad x+y+z = 10u + 5w$

So $\text{Mass}(D_2^+) = \iiint_S (1 + 10u + 5w)\,\frac{125}{2}\,dV = \frac{125}{2}\left(\iiint_S 1\,dV + 10\iiint_S u\,dV + 5\iiint_S w\,dV\right)$

S is symmetric in u & w

$= \frac{125}{2}\left(\frac{1}{6} + (10+5)\,\frac{1}{24}\right) = \frac{125}{48}(19) =$

Conclusion: $\text{Mass} = \frac{125\cdot 3}{48} + \frac{875}{12} + \frac{19\cdot 125}{48} = \frac{6250}{48} = \boxed{\frac{3125}{24}}$

b) How to compute the centroid? $y\text{-comp} = \frac{M_{xz}}{\text{Total mass}} = \iiint_D y\cdot\rho(x,y,z)\,dV$

We break D into the 3 pieces we did before & use the 3 changes of variables.

D_1^-: $\iiint_{D_1^-} y\,\rho(x,y,z)\,dV = \iiint_S \left(5v + \frac{5}{3}w\right)(5u+1)\,\frac{125}{6}\,dV$

$= \frac{125}{6}\iiint_S 25uv + 5v + \frac{25}{3}uw + \frac{5}{3}w\,dV\,)$

$= \frac{125}{6}\left(\,25\underbrace{\iiint_S uv\,dV} + \frac{25}{3}\underbrace{\iiint_S uw\,dV} + 5\underbrace{\iiint_S v\,dV} + \frac{5}{3}\underbrace{\iiint_S w\,dV}\right.$

same.

same & $= 1/24$

We compute $\iiint_S uv\,dV = \int_0^1\int_0^{1-u}\int_0^{1-u-v} uv\,dw\,dv\,du = \int_0^1\int_0^{1-u} uv(1-u-v)\,dv\,du$

$$= \int_0^1\int_0^{1-u} uv - u^2v - uv^2\,dv\,du = \int_0^1 \frac{u(1-u)^2}{2} - \frac{u^2(1-u)^2}{2} - \frac{u(1-u)^3}{3}\,du = \boxed{1/120}$$

Then $\iiint_{D_1^-} y\,\rho(x,y,z)\,dV = \frac{125}{6}\left(\left(25+\frac{25}{3}\right)\left(\frac{1}{120}\right) + \frac{1}{24}\left(5+5/3\right)\right) = \frac{125}{6}\left(\frac{100}{3\cdot120}+\frac{20}{3\cdot 24}\right)$

$$= \frac{200\cdot125}{18\cdot120} = \boxed{\frac{625}{54}}$$

Similarly:

D_1^+: $\iiint_{D_1^+} y\,\rho(x,y,z)\,dV = \iiint_S 125\left(5u+\frac52 v+\frac53 w\right)\cdot(10u+1)\,dV$

$$=125\iiint_S\left(50u^2+5u+\frac{50}{2}uv+\frac52 v+\frac{50}{3}uw+\frac53 w\right)dV$$

$$=125\Big(50\underbrace{\iiint_S u^2\,dV}_{?} + \underbrace{\left(5+\frac52+\frac53\right)}_{=\frac{55}{6}}\underbrace{\iiint_S u\,dV}_{=1/24} + \underbrace{\left(\frac{50}{2}+\frac{50}{3}\right)}_{=\frac{5\cdot50}{6}}\underbrace{\iiint_S uv\,dV}_{1/120}$$

Then: $\iiint_S u^2\,dV = \int_0^1\int_0^{1-u}\int_0^{1-u-v}u^2\,dw\,dv\,du = \int_0^1\int_0^{1-u}u^2(1-u-v)\,dv\,du = \frac{1}{60}$

$$\Rightarrow \iiint_{D_1^+} y\,\rho(x,y,z) = 125\left(\frac{50}{60}+\frac{55}{6\cdot24}+\frac{5\cdot50}{6\cdot120}\right) = 125\cdot\frac{25}{16} = \boxed{\frac{3125}{16}}$$

- D_2^+: $\iiint_{D_2^+} y\,\rho(x,y,z)\,dV = \iiint_S \frac{125}{2}\left(5u+\frac52 v\right)(10u+1)\,dV$

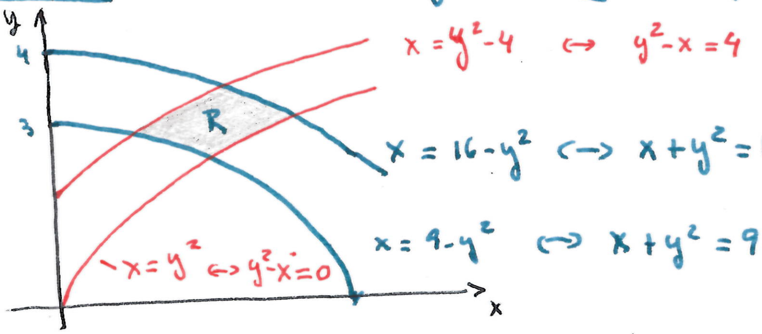
$$=\frac{125}{2}\left(50\iiint_S u^2\,dV + \left(5+\frac52\right)\iiint_S u\,dV + \frac{50}{2}\iiint uv\,dV\right)$$

$$=\frac{125}{2}\left(50\,\frac{1}{60}+\frac{15}{2}\,\frac{1}{24}+\frac{50}{2}\,\frac{1}{120}\right) = \frac{125}{2}\,\frac{15}{8} = \boxed{\frac{1875}{16}}$$

Conclusion: $M_y = \frac{625}{54}+\frac{3125}{16}+\frac{1875}{16} = 625\cdot\frac{14}{17} = \boxed{\frac{8750}{17}}$

y-component: $\frac{M_y}{\text{mass}} = \frac{8750/17}{3125/24} = \frac{210000}{53125} = \boxed{\frac{336}{85}}$

Problem 2: We start by drawing the region:



$\boxed{y \geq 0}$

- We propose the following change of coordinates

$$\begin{cases} u = y^2 - x \\ v = x + y^2 \end{cases} \quad \text{so } R \text{ comes from the rectangle } \boxed{\begin{matrix} 0 \leq u \leq 4 \\ 9 \leq v \leq 16 \end{matrix}} \text{ in } (u,v)\text{-coords.}$$

We find.

$$T: \begin{cases} x = \dfrac{v-u}{2} \\ y = \sqrt{u + \dfrac{v-u}{2}} = \sqrt{\dfrac{u+v}{2}} \end{cases} \longrightarrow J(u,v) = \begin{array}{c|cc} & \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \\ \hline x & -\frac{1}{2} & \frac{1}{2} \\ y & \frac{(u+v)^{-1/2}}{2\sqrt{2}} & \frac{(u+v)^{-1/2}}{2\sqrt{2}} \end{array}$$

$$= -\frac{1}{2}\left(\frac{(u+v)^{-1/2}}{2\sqrt{2}}\right) - \frac{1}{2}\frac{(u+v)^{-1/2}}{2\sqrt{2}}$$

$$= \frac{-1}{2\sqrt{2}\sqrt{u+v}}$$

$y^2 = \frac{u+v}{2}$

$$\Rightarrow \iint_R y^2 \, dV = \int_0^4 \int_9^{16} \frac{u+v}{2} \frac{1}{2\sqrt{2}\sqrt{u+v}} \, dv\, du$$

$$= \frac{1}{4\sqrt{2}} \int_0^4 \int_9^{16} \sqrt{u+v} \, dv\, du = \frac{1}{4\sqrt{2}} \int_0^4 \frac{(u+v)^{3/2}}{3/2} \Bigg|_{v=9}^{v=16} du = \frac{1}{6\sqrt{2}} \int_0^4 (u+16)^{3/2} - (u+9)^{3/2} \, du$$

$$= \frac{1}{6\sqrt{2}} \frac{(u+16)^{5/2} - (u+9)^{5/2}}{5/2} \Bigg|_{u=0}^{u=4} = \frac{1}{15\sqrt{2}} \left(20^{5/2} - (13)^{5/2} - 16^{5/2} + 9^{5/2} \right)$$

$$= \boxed{\frac{\sqrt{2}}{30} \left(32 \cdot 5^{5/2} - 13^{5/2} - 781 \right)}$$

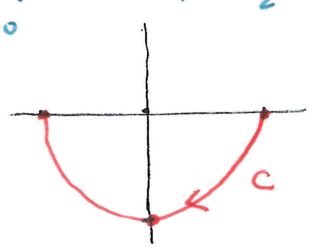
Problem 3: (a) $\int_C xy \, ds = ?$ Parametriz: $C: r: [0,1] \longrightarrow \mathbb{R}^2$

$$t \longmapsto (1,2) + t\big((3,5)-(1,2)\big)$$
$$= (1,2) + t(2,3) = (1+2t, 2+3t)$$

By def $\int_0^1 (1+2t)(2+3t) \, |\langle 2,3 \rangle| \, dt = \int_0^1 (2+7t+6t^2)\sqrt{13}\, dt$

$$= \sqrt{13} \left(2t + \frac{7t^2}{2} + 2t^3 \Big|_{t=0}^{t=1} \right) = \sqrt{13}\left(2 + \frac{7}{2} + 2\right) = \boxed{\frac{\sqrt{13}\, 15}{2}}$$

(b) We draw the curve



Parametriz: $C^{op}: r: [-\pi, 0] \longrightarrow \mathbb{R}^2$

$$\theta \longmapsto \langle \cos\theta, \sin\theta \rangle$$

$\vec{r}'(\theta) = \langle -\sin\theta, \cos\theta \rangle \Rightarrow |\vec{r}'(\theta)| = 1$

$$\int_C xy\,ds = -\int_{C^{op}} xy\,ds = -\int_{-\pi}^{0} (\cos\theta \sin\theta)\,1\,d\theta = \frac{-1}{2}\int_{-\pi}^{0} \underbrace{2\sin\theta\cos\theta}_{=\sin 2\theta}\, d\theta$$

$$= \frac{-1}{2}(-\cos 2\theta)\Big|_{\theta=-\pi}^{\theta=0} = \frac{1}{2}\left(\cos 0 - \cos(-2\pi)\right) = \frac{1}{2}(1-1) = \boxed{0}$$

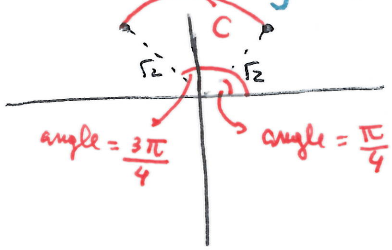
Problem 4: (a) $\vec{F}(x,y) = \langle \frac{1}{y}, 0\rangle$ We curve from $(1,1)$ to $(-1,1)$ has the linear parameterization

$$\vec{r}:[0,1] \longrightarrow \mathbb{R}^2 \qquad \Rightarrow \vec{r}\,'(t) = \langle -2, 0\rangle \Rightarrow \boxed{\vec{T} = \langle -2, 0\rangle}$$

$$t \longmapsto (1,1) + t\left((-1,1)-(1,1)\right) = (1,1) + t(-2,0) = (1-2t, 1)$$

$$\text{Work} = \int_C \vec{F}\cdot T\,ds = \int_0^1 \langle \tfrac{1}{1}, 0\rangle \cdot \langle -2, 0\rangle\,dt = \int_0^1 -2\,dt = \boxed{-2}$$

(b) The circle containing $(1,1)$ & $(-1,1)$ with radius $\sqrt{2}$



$$C \text{ curve } \vec{r}: \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \longrightarrow \mathbb{R}^2 \qquad \Rightarrow \vec{r}\,'(\theta) = \langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta\rangle \Rightarrow \vec{T}(\theta) = \langle -\sin\theta, \cos\theta\rangle$$

$$\theta \longmapsto \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta\rangle$$

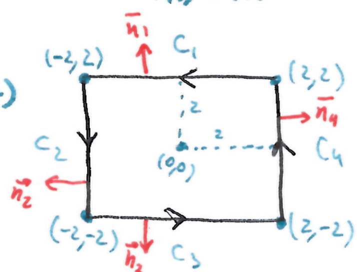
$$\text{Work} = \int_C F\cdot T\,ds = \sqrt{2}\int_{\pi/4}^{3\pi/4} \langle \frac{1}{\sqrt{2}\sin\theta}, 0\rangle \langle -\sin\theta, \cos\theta\rangle\,d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \frac{-\sin\theta}{\sin\theta}\,d\theta = \frac{-1}{\sqrt{2}}\sqrt{2}\,\theta\Big|_{\pi/4}^{3\pi/4} = -\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \boxed{-\frac{\pi}{2}}$$

Problem 5: $\vec{F}(x,y) = \langle x, y\rangle$ Recall: flux $\int_C \vec{F}\cdot\vec{n}\,ds$ $\vec{n}$ = outer normal

$$\vec{r} = \langle x(t), y(t)\rangle \Rightarrow \vec{n} = \frac{\langle \vec{y}\,'(t), \vec{x}\,'(t)\rangle}{|\vec{r}\,'(t)|}$$

(a)



We break the curve into 4 pieces

$$\text{Flux}_{(C)} = \text{Flux}(C_1) + \text{Flux}(C_2) + \text{Flux}(C_3) + \text{Flux}(C_4)$$

We parameterized these 4 curves:

C_1^{op}: $\vec{r}_1: [-2,2] \longrightarrow \mathbb{R}^2$ $\vec{r}_1(t) = \langle t, 2\rangle$ $\rightsquigarrow$ $\vec{n}_1 = \langle 0, 1\rangle$, $|\vec{r}_1\,'(t)| = |\langle 1, 0\rangle| = 1$

C_2^{op}: $\vec{r}_2: [-2,2] \longrightarrow \mathbb{R}^2$ $\vec{r}_2(t) = \langle -2, t\rangle$ $\rightsquigarrow$ $\vec{n}_2 = \langle -1, 0\rangle$, $|\vec{r}_2(t)| = |\langle 0, 1\rangle| = 1$

C_3: $\vec{r}_3: [-2,2] \longrightarrow \mathbb{R}^2$ $\vec{r}_3(t) = \langle t, -2\rangle$ $\rightsquigarrow$ $\vec{n}_3 = \langle 0, -1\rangle$, $|\vec{r}_3(t)| = |\langle 1, 0\rangle| = 1$

C_4: $\vec{r}_4: [-2,2] \longrightarrow \mathbb{R}^2$ $\vec{r}_4(t) = \langle 2, t\rangle$ $\rightsquigarrow$ $\vec{n}_4 = \langle 1, 0\rangle$, $|\vec{r}_4(t)| = |\langle 0, 1\rangle| = 1$

$$\text{Flux}(C_1) = -\text{Flux}(C_1^{op}) = -\int_{-2}^{2} \langle t, 2\rangle \cdot \langle 0,1\rangle\, 1\, dt = -\int_{-2}^{2} 2\, dt = -2\cdot 4 = \boxed{-8}$$

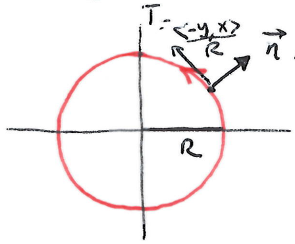
$$\text{Flux}(C_2) = -\text{Flux}(C_2^{op}) = -\int_{-2}^{2} \langle -2, t\rangle \cdot \langle -1, 0\rangle\, 1\, dt = -\int_{-2}^{2} 2\, dt = -2\cdot 4 = \boxed{-8}$$

$$\text{Flux}(C_3) = \int_{-2}^{2} \langle t,-2\rangle \cdot \langle 0,-1\rangle\, dt = \int_{-2}^{2} 2\, dt = \boxed{8}$$

$$\text{Flux}(C_4) = \int_{-2}^{2} \langle 2,t\rangle \cdot \langle 1,0\rangle\, dt = \int_{-2}^{2} 2\, dt = \boxed{8}$$

Conclusion: $\text{Flux}(C) = 0$

(b)



Param. of C: $\vec{r}(\theta) = \langle R\cos\theta, R\sin\theta\rangle \quad 0 \le \theta \le 2\pi$. $\quad |r'(\theta)| = R$

$$\int_C F\cdot\vec{n}\, ds = \int_0^{2\pi} \langle R\cos\theta, R\sin\theta\rangle \cdot \frac{\langle R\cos\theta, R\sin\theta\rangle}{R}\, R\, d\theta$$

$$= \int_0^{2\pi} R^2(\cos^2\theta + \sin^2\theta)\, d\theta = \int_0^{2\pi} R^2\, d\theta = \boxed{2\pi R^2}$$

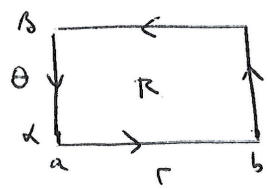
Problem 6: (a) We ~~recall~~ the circulation is $\int_C \vec{F}\cdot\vec{T}\, ds$

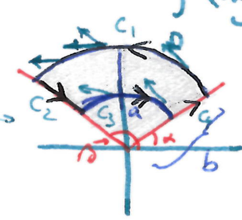
C circle param $\vec{r}: [0, 2\pi] \longrightarrow \mathbb{R}^2$ $\qquad \vec{T} = \langle -\sin\theta, \cos\theta\rangle$

$\theta \longmapsto \langle r\cos\theta, r\sin\theta\rangle \qquad |\vec{r}'| = r$

$$\text{Circ} = \int_0^{2\pi} \langle -r\sin\theta, r\cos\theta\rangle \cdot \langle -\sin\theta, \cos\theta\rangle\, r\, d\theta = r^2 \int_0^{2\pi} \underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}\, d\theta = \boxed{2\pi r^2}$$

(b) Polar rectangle:





(i) We know the circulation on the 2 arcs have opposite signs: On C_2 & C_4 $\vec{F} \perp \vec{T}$
but the outer ~~arc~~ will have higher circulation, so the expectation is a POSITIVE circ.

(ii) ~~Case~~ C_1: $\vec{r}_1 : [\alpha, \beta] \longrightarrow \mathbb{R}^2$ $\qquad \vec{r}_1(\theta) = \langle b\cos\theta, b\sin\theta\rangle \rightsquigarrow T_1(\theta) = \frac{\langle -b\sin\theta, b\cos\theta\rangle}{b}$

$\vec{F} = b\vec{T}_1$

C_3^{op}: $\vec{r}_3 : [\alpha, \beta] \longrightarrow \mathbb{R}^2$ $\qquad \vec{r}_3(\theta) = \langle a\cos\theta, a\sin\theta\rangle \rightsquigarrow T_3(\theta) = \frac{\langle -a\sin\theta, a\cos\theta\rangle}{a}$

$\vec{F} = a\vec{T}_3$

C_2^{op}: $\vec{r}_2 : [a, b] \longrightarrow \mathbb{R}^2$ $\qquad \vec{r}_2(t) = \langle t\cos\beta, t\sin\beta\rangle \rightsquigarrow T_2(t) = \langle \cos\beta, \sin\beta\rangle$

$\vec{F} \perp \vec{T}_2$

C_4: $\vec{r}_4 : [a, b] \longrightarrow \mathbb{R}^2$ $\qquad \vec{r}_4(t) = \langle t\cos\alpha, t\sin\alpha\rangle \rightsquigarrow T_4(t) = \langle \cos\alpha, \sin\alpha\rangle$

$\vec{F} \perp \vec{T}_4$

$\Rightarrow \text{Circ}(C_4) = \int 0\, ds = 0 = \text{Circ}(C_2)$

$$\mathrm{circ}(C_3) = -\mathrm{circ}(C_3^{op}) = -\int_\alpha^\beta a \cdot ds = -\int_\alpha^\beta a(a\,dt) = -a^2\int_\alpha^\beta dt = -a^2(\beta-\alpha).$$

$$\mathrm{circ}(C_1) = \int_\alpha^\beta b \cdot ds = \int_\alpha^\beta b^2\,dt = b^2(\beta-\alpha)$$

$$\text{Total} = \mathrm{circ}(C) = 0+0+b^2(\beta-\alpha) + (-a^2(\beta-\alpha)) = \underbrace{(b-a)}_{>0}\underbrace{(b+a)}_{>0}\underbrace{(\beta-\alpha)}_{>0}$$

$(0<a<b)$

(iii) NOTE: $F \perp T$ on C_2 & C_4, and $\vec{F} = \dfrac{\vec{T}_1}{b^p}$ on C_1

$\vec{F} = \dfrac{\vec{T}_2}{a^p}$ on C_3^{op}.

$$\begin{cases} \mathrm{circ}(C_1) = \displaystyle\int_\alpha^\beta \frac{1}{b^p}\, b\,dt = b^{1-p}(\beta-\alpha) \\ \mathrm{circ}(C_3) = -\displaystyle\int_\alpha^\beta \frac{1}{a^p}\, a\,dt = -a^{1-p}(\beta-\alpha) \end{cases}$$

$$\Rightarrow \mathrm{circ}(C) = \underbrace{(\beta-\alpha)}_{>0}(b^{1-p}-a^{1-p})$$

$\searrow 0$ only if $p=1$ because $0<a<b$.

The circulation is 0 if and only if $p=1$.

Problem 7: (a) We use the conservative Test because $\mathbb{R}^2$ is conn. & simply connected

$f = 3x^2y^2 \rightarrow f_y = 6x^2y$

$g = 2x^3y \Rightarrow g_x = 6x^2y$ (=) so F is conservative.

We find φ by integrating $\varphi_x = f = 3x^2y^2 \Rightarrow \varphi_{(x,y)} = \int 3x^2y^2\,dx + C_{(y)}$

$= x^3y^2 + C_{(y)}$

Then $2x^3y = g = \varphi_y = \frac{\partial}{\partial y}(x^3y^2) + C'(y) = 2x^3y + C'(y)$

$\Rightarrow C'(y) = 0$ so $C_{(y)}$ is constant

Potential: $\boxed{\varphi_{(x,y)} = x^3y^2 + C.}$

(b) Use the test because the region $\{x>0\}$ is conn & simply connected.

$f = \dfrac{-y}{x^2+y^2} \Rightarrow f_y = -\dfrac{(x^2+y^2)-(-y)2y}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ (=) so F is conservative

$g = \dfrac{x}{x^2+y^2} \Rightarrow g_x = \dfrac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

We find φ by integrating $\varphi_{(x,y)} = \int \frac{-y}{x^2+y^2}\,dx = -y\int \frac{1}{x^2+y^2}\,dx + C_{(y)}$

$$\int \frac{1}{x^2+y^2}\,dx = \frac{1}{y^2}\int \frac{1}{1+\left(\frac{x}{y}\right)^2}\,dx \underset{u=\frac{x}{y}}{=} \frac{1}{y}\int \frac{1}{1+u^2}\,du = \frac{1}{y}\tan^{-1}(u) = \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)$$

$$\Rightarrow \varphi_{y\,(x,y)} = \frac{d}{dy}\left(-\tan^{-1}\left(\frac{x}{y}\right)\right) + C'(y) \Rightarrow \frac{-x}{x^2+y^2} = \left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\right)\cdot\frac{-x}{y^2} + C'_{(y)}$$

$$\frac{-x}{x^2+y^2} = \frac{-x}{(y^2+x^2)} + C'(y)$$ so $C'(y) = 0$ & we conclude

$$\boxed{\varphi_{(x,y)} = -\tan^{-1}\left(\frac{x}{y}\right) + C}$$

(c) Use the test (the region $\mathbb{R}^3$ is conn & simply connected)

$f = y^2z^3 \Rightarrow f_y = 2yz^3$ & $f_z = 3y^2z^2$

$g = 2xyz^3 + 6yz \Rightarrow g_x = 2yz^3$ & $g_z = 6xyz^2 + 6y$

$h = 3xy^2z^2 + 3y^2 \Rightarrow h_x = 3y^2z^2$ & $h_y = 6xyz^2 + 6y$

$f_y = g_x$, $f_z = h_x$, $g_z = h_y$ so $\vec{F}$ is conservative

We find the potential:

$$\varphi_{(x,y,z)} = \int y^2z^3\,dx + C_{(y,z)} = y^2z^3x + C_{(y,z)}$$

$$\Rightarrow \varphi_y = 2yz^3x + C_y(y,z) = 2xyz^3 + 6yz \Rightarrow C_y(y,z) = 6yz$$

so $C_{(y,z)} = \int 6yz\,dy + C_{(z)} = \boxed{3y^2z + C_{(z)}}$

$$\varphi_z = 3y^2z^2x + 3y^2 + C'_{(z)} = 3xy^2z^2 + 3y^2 \Rightarrow C'_{(z)} = 0$$ so $C_{(z)} = C$.

Conclusion: $\boxed{\varphi_{(x,y)} = y^2z^3x + 3y^2z + C}$