Recitation XI

Problem 1: In all the 3 examples, the region where $\vec{F}$ is defined is connected & simply connected. Then: $\vec{F}$ is conservative $\underset{\text{in } \mathbb{R}^2}{}$ $\Longleftrightarrow$ $\text{curl}(F) = 0$.

(a) $f = 3x^2y^2 \Rightarrow f_y = 6x^2y$

$g = 2x^3y \Rightarrow g_x = 6x^2y$ $\quad$ $\shortparallel$ so $\text{curl} = g_x - f_y = 0$ ✓

We find the potential function φ by integrating: $\varphi_x = f = 3x^2y^2$

$$\Rightarrow \varphi(x,y) = \int 3x^2y^2\,dx + C(y) = x^3y^2 + C(y)$$

We compute $\frac{\partial \varphi}{\partial y}$ to get the function $C(y)$:

$$2x^3y = g = \varphi_y = \frac{\partial}{\partial y}(x^3y^2) + C'(y) = 2x^3y + C'(y) \Rightarrow C'(y) = 0$$

so $C(y) = C$ const

$$\Rightarrow \boxed{\varphi(x,y) = x^3y^2 + C}$$

(b) $f = \frac{-y}{x^2+y^2} \Rightarrow f_y = \frac{-(x^2+y^2) - (-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad$ $\shortparallel$ so $\text{curl} = g_x - f_y = 0$

$g = \frac{x}{x^2+y^2} \Rightarrow g_x = \frac{(x^2+y^2) - x2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

We integrate f: $\varphi(x,y) = \int \frac{-y}{x^2+y^2}\,dx = -y \int \frac{1}{x^2+y^2}\,dx + C(y)$

$$\int \frac{1}{x^2+y^2}\,dx = \frac{1}{y^2}\int \frac{1}{\left(\frac{x}{y}\right)^2+1}\,dx \underset{u=\frac{x}{y}}{=} \frac{1}{y}\int \frac{1}{u^2+1}\,du = \frac{1}{y}\tan^{-1}(u) = \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)$$

So $\underset{\frac{x}{y^2+x^2} = g =}{\varphi_y(x,y)} = \frac{d}{dy}\left(-\tan^{-1}\left(\frac{x}{y}\right)\right) + C'(y) = \frac{-1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{-x}{y^2}\right) + C'(y) = \frac{+x}{y^2+x^2} + C'(y)$

$\Rightarrow C'(y) = 0$ so $C(y) = C$ constant

$$\Rightarrow \boxed{\varphi(x,y) = -\tan^{-1}\left(\frac{x}{y}\right) + C}$$

(c) $\begin{cases} f = y^2z^3 \\ g = 2xyz^3 + 6yz \\ h = 3xy^2z^2 + 3y^2 \end{cases}$ $\begin{matrix} \Rightarrow & f_y = 2yz^3 \ \& \ f_z = 3y^2z^2 \\ \Rightarrow & g_x = 2yz^3 \ \& \ g_z = 6xyz^2 + 6y \\ \Rightarrow & h_x = 3x^2z^2 \ \& \ h_y = 6xyz^2 + 6y \end{matrix}$

So $f_y = g_x$, $f_z = h_x$, $g_z = h_y$ so $\vec{F}$ is conservative (b/c $\mathbb{R}^3$ is conn & simply connected)

We find the potential by integrating:

$$\varphi_{(x,y,z)} = \int y^2z^3\,dx + C_{(y,z)} = y^2z^3x + C_{(y,z)}$$

$$\Rightarrow g = \varphi_y = 2yz^3x + C_y(y,z) = 2xyz^3 + 6yz \Rightarrow C_y(y,z) = 6yz$$

so $C_{(y,z)} = \int 6yz\,dy + C_{(z)} = \boxed{3y^2z + C_{(z)}}$

Differentiate one more time:

$$h = \varphi_z = 3y^2z^2x + 3y^2 + C'_{(z)} = 3xy^2z^2 + 3y^2 \Rightarrow C'_{(z)} = 0 \text{ so } C_{(z)} = C$$

$$\Rightarrow \boxed{\varphi(x,y) = y^2z^3x + 3y^2z + C}$$

<u>Problem 2:</u> (a) By definition $\operatorname{div}(F) = f_x + g_y = \frac{\partial\psi}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial\psi}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)$

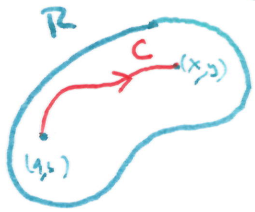
$= \psi_{yx} - \psi_{xy} = 0$

↳ $\psi_{yx} = \psi_{xy}$ because they are cont.

(b) We define Ψ by an integral:

$$\Psi_{(x,y)} = \int_{C_{(x,y)}} \vec{F}_{(x,y)} \cdot \vec{n}\,ds$$

where $C_{(x,y)}$ is a piecewise smooth curve in R joining a fixed pt (a,b) to a point (x,y). Such a curve exists because R is connected.



Why is $\Psi_{(x,y)}$ well-defined?

If $C'_{(x,y)}$ is another curve from (a,b) to (x,y), then we let $\tilde{C} = C + C'^{op}$ be ~~the~~ resulting closed loop from (a,b) to (x,y).

We assume $\tilde{C}$ is simple (*)

Then the region $\tilde{R}$ enclosed by $\tilde{C}$ is connected & simply connected. By Green's Thm

$$\oint_{\tilde{C}} F \cdot \vec{n}\,ds = \iint_{\tilde{R}} \underbrace{\operatorname{div}(F)}_{=0}\,dA = 0 \quad \Rightarrow 0 = \int_C F\cdot\vec{n}\,ds - \int_{C'} F\cdot\vec{n}\,ds$$

$$\Rightarrow \boxed{\int_C F\cdot\vec{n}\,ds = \int_{C'} F\cdot\vec{n}\,ds}$$

(*) Otherwise, we break $\tilde{C}$ into simple loops =  $\begin{cases} C = C_1 + C_2 \\ C' = C_1' + C_2' \end{cases}$

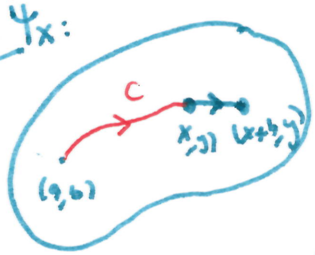
Notice $F\cdot\vec{\eta}\,ds = \tilde{F}\cdot T\,ds$ for $\tilde{F} = \langle -g, +f\rangle$, because $T = \dfrac{\langle x'(t), y'(t)\rangle}{|\vec{r}\,'(t)|}$

so $\vec{\eta} = \dfrac{\langle y'(t), -x'(t)\rangle}{|\vec{r}\,'(t)|}$ $\quad F\cdot\vec{\eta}\,ds = f(\vec{r}(t))\,y'(t) + g(\vec{r}(t))\cdot(-x'(t)) = \tilde{F}\cdot\vec{T}\,ds$

Then $\oint_C \tilde{F}\cdot\vec{T}\,ds = \oint_C F\cdot\vec{\eta}\,ds$ so $\tilde{F}$ is path independent.

We show that $\nabla\Psi = \langle \Psi_x, \Psi_y\rangle = \langle -g, f\rangle = \tilde{F}$ using the definition of partials

For Ψ_x:



we take the path = C + segment joining (x,y) & (x+h,y) as the path $\tilde{C}$ from (a,b) to (x+h,y)

$$\text{So } \Psi(x+h,y) = \int_{\tilde{C}} F\cdot\vec{\eta}\,ds = \int_C F\cdot\vec{\eta}\,ds + \int_{(x,y)-(x+h,y)} F\cdot\vec{\eta}\,ds = \Psi(x,y) + \int_{(x,y)-(x+h,y)} F\cdot\vec{\eta}\,ds$$

$$\text{so } (*) = \frac{\Psi(x+h,y)-\Psi(x,y)}{h} = \frac{1}{h}\int_{(x,y)-(x+h,y)} F\cdot\vec{\eta}\,ds = \frac{1}{h}\int_0^h \langle f(x+t,y), g(x+t,y)\rangle\cdot\langle 0,-1\rangle\,dt$$

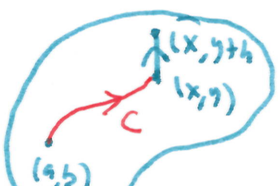
$\vec{r}(t) = \langle x+t, y\rangle$, $0\le t\le h$ $\Rightarrow \vec{r}\,'(t) = \langle 1,0\rangle \Rightarrow \vec{\eta} = \langle 0,-1\rangle$

$$= \frac{1}{h}\int_0^h -g(x+t,y)\,dt = -\frac{1}{h}\left(\int_0^h g(x+t,y)\,dt - \underbrace{\int_0^0 g(x+t,y)\,dt}_{=0}\right)$$

$$\Psi_x = \lim_{h\to 0}((*)) \overset{\text{def}}{=} -\frac{d}{dh}\int_0^h g(x+t,y)\,dt\Big|_{h=0} \underset{\text{Fund. Thm of calculus}}{=} -g(x+h,y)\Big|_{h=0} = -g(x,y).$$

For Ψ_y

Similarly, we take the path C followed by the vertical segment from (x,y) to (x, y+h) as the path $\tilde{C}$ from (a,b) to (x,y+h)



$$\text{So } \Psi(x,y+h) = \int_{\tilde{C}} F\cdot\vec{\eta}\,ds = \int_C F\cdot\vec{\eta}\,ds + \int_{(x,y)\text{ to }(x,y+h)} F\cdot\vec{\eta}\,ds = \Psi(x,y) + \int_{(x,y)\text{ to }(x,y+h)} F\cdot\vec{\eta}\,ds$$

$$\text{so } \frac{\Psi(x,y+h)-\Psi(x,y)}{h} = \frac{1}{h}\int_{(x,y)} F\cdot\vec{\eta}\,ds = \frac{1}{h}\int_0^h \langle f(x,y+t), g(x,y+t)\rangle\cdot\langle 1,0\rangle\,dt$$

$$= \frac{1}{h}\left(\int_0^h f(x,y+t)\,dt - \int_0^0 f(x,y+t)\,dt\right)$$

$\vec{r}(t) = \langle x, y+t\rangle$, $0\le t\le h$ $\Rightarrow \vec{r}\,'(t) = \langle 0,1\rangle \Rightarrow \vec{\eta} = \langle 1,0\rangle$

So $\Psi_y = \lim_{h\to 0} \dfrac{\Psi(x+y+h) - \Psi(x,y)}{h} = \dfrac{d}{dh}\left(\int_0^h f(x,y+t)\,dt\right)\Big|_{h=0} = f(x,y+0) = f(x,y)$

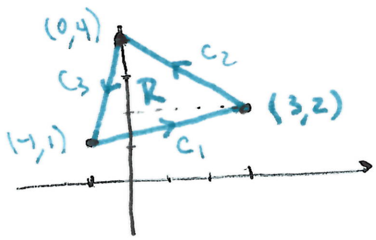
Conclusion $\Psi_x = -f$, & $\Psi_y = g$ as we wanted to show.

(c) We give a parameterization of a fixed level curve $\Psi(x,y)=c$ as $\langle x(t), y(t)\rangle$ & use implicit differentiation.

$$0 = \frac{d}{dt}\big(\underbrace{\Psi(x(t),y(t))}_{=\text{constant}}\big) \underset{\text{Chain Rule}}{=} \frac{\partial\Psi}{\partial x}\cdot x' + \frac{\partial \Psi}{\partial y}\cdot y' = \underbrace{\left\langle \frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right\rangle}_{=\vec F}\cdot \underbrace{\langle y', -x'\rangle}_{\substack{\text{outer normal}\\ \text{(non-unit)}}}$$

so $\vec F \perp \vec\eta$ (unit outer normal). Then, $\vec F$ is tangent to the flow curves. (v. field in $\mathbb{R}^2$).

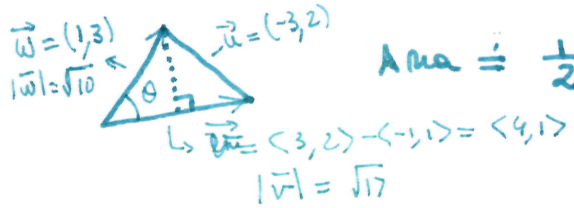
Problem 3: (a) We start by drawing the triangle:



The interior of the triangle is connected & simply connected

$$\text{Cir}(F) = \oint_C F\cdot\vec T\,ds \underset{\text{Green's Thm.}}{=} \iint_R \text{curl}(F)\,dA = \iint_R 1\,dA = \text{Area}(R)$$

curl $= g_x - f_y = 0 - (-1) = 1$



$$\text{Area} = \frac12 |\vec v \times \vec w| = \frac12 \left| \begin{vmatrix} i & j & k\\ 4 & 1 & 0\\ 1 & 3 & 0\end{vmatrix}\right| = \frac12 |\vec k\,(12-1)| = \boxed{\frac{11}{2}}$$

So $\boxed{\text{circulation} = 11/2}$

(b) Use the parameterization of the 3 curves:

C_1: $\vec r_1(t): [0,1] \longrightarrow \mathbb{R}^2$ $\quad \vec r_1(t) = \langle -1,1\rangle + t\langle 4,1\rangle = \langle -1+4t, 1+t\rangle \Rightarrow \vec r_1'(t) = \langle 4,1\rangle$

C_2: $r_2(t): [0,1] \longrightarrow \mathbb{R}^2$ $\quad \vec r_2(t) = \langle 3,2\rangle + t\langle -3,2\rangle = \langle 3-3t, 2+2t\rangle \Rightarrow \vec r_2'(t) = \langle -3, 2\rangle$

C_3^{op}: $r_3(t): [0,1] \longrightarrow \mathbb{R}^2$ $\quad \vec r_3(t) = \langle -1,1\rangle + t\langle 1,3\rangle = \langle -1+t, 1+3t\rangle \Rightarrow r_3'(t) = \langle 1,3\rangle$

$$\oint_C \vec F\cdot\vec T\,ds = \int_{C_1} \vec F\cdot\vec T\,ds + \int_{C_2} \vec F\cdot\vec T\,ds - \int_{C_3^{op}} \vec F\cdot\vec T\,ds$$

$$\int_{C_1} \vec F\cdot\vec T\,ds = \int_0^1 \langle -(1+t), 0\rangle\cdot\langle 4,1\rangle\,dt = -4\int_0^1 1+t\,dt = -4\left(t+\frac{t^2}{2}\right)\Big|_{t=0}^{t=1} = -4\,\frac32 = \boxed{-6}$$

$$\int_{C_2} \vec F\cdot\vec T\,ds = \int_0^1 \langle -(2+2t), 0\rangle\cdot\langle -3,2\rangle\,dt = +6\int_0^1 1+t\,dt = -6\cdot\frac32 = \boxed{+9}$$

$$\int_{C_3^{op}} \vec{F}\cdot\vec{T}\, ds = \int_0^1 -\langle 1+3t, 0\rangle \cdot \langle 1,3\rangle\, dt = -\int_0^1 1+3t\, dt = -\left(t+\frac{3}{2}t^2\right)\Big|_{t=0}^{t=1}$$

$$= -\left(1+\frac{3}{2}\right) = -\frac{5}{2}.$$

$$\Rightarrow \text{Circulation} = -6+9-\left(-\frac{5}{2}\right) = 3+\frac{5}{2} = \boxed{\frac{11}{2}}$$

(c) $\vec{F} = \langle -y, 0\rangle \rightsquigarrow f_y = -1$ & $g_x = 0$ so $\text{curl}(F) \neq 0$ & $\mathbb{R}^2$ is connected & simply connected so $\vec{F}$ is not path independent because it's not conservative.

Also: the boundary of the triangle shows that $\int_{C_3^{op}} \vec{F}\cdot d\vec{r} \neq \int_{C_1+C_2} \vec{F}\cdot dr$, so $\vec{F}$ is not path independent.

Problem 4:

We can solve the problem in 2 ways. We can use the formula for computing N = the unit INNER normal vector in the TNB frame.

$$\vec{r}(t) = \langle \cos t, \sin t, \cos t\rangle \Longrightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t, -\sin t\rangle$$

$$\Rightarrow \vec{T}(t) = \frac{\langle -\sin t, \cos t, -\sin t\rangle}{\sqrt{\cos^2 t + 2\sin^2 t}} = \frac{\langle -\sin t, \cos t, -\sin t\rangle}{\sqrt{1+\sin^2 t}}$$

So $\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}(t)}{\left|\frac{d\vec{T}}{dt}(t)\right|}$ $\Rightarrow \dfrac{\langle -\cos t, -\sin t, -\cos t\rangle}{\sqrt{1+\sin^2 t}} + \dfrac{\langle \sin^2 t\cos t, -\cos^2 t\sin t, \sin^2 t\cos t\rangle}{\left(\sqrt{1+\sin^2 t}\right)^3} = \dfrac{dT}{dt}(t)$

$$\frac{dT}{dt} = \frac{\langle \cos t\,(\sin^2 t - (1+\sin^2 t)), -\sin t\,(\cos^2 t + (1+\sin^2 t)), \cos t\,(\sin^2 t - (1+\sin^2 t))\rangle}{(1+\sin^2 t)^{3/2}}$$

$$= \frac{\langle -\cos t, -2\sin t, -\cos t\rangle}{(1+\sin^2 t)^{3/2}}$$

$$\Rightarrow \vec{n}(t) = -\frac{\langle -\cos t, -2\sin t, -\cos t\rangle}{\sqrt{2\cos^2 t + 4\sin^2 t}}$$ is outer unit normal.

$$\text{Circ}(F) = \oint_C F\cdot T\, ds = \int_0^{2\pi} \langle \cos t, \cos t, -\sin t\rangle \cdot \langle -\sin t, \cos t, -\sin t\rangle\, dt$$

$$= \int_0^{2\pi} -\cos t\sin t + \underbrace{\cos^2 t + \sin^2 t}_{=1}\, dt = \int_0^{2\pi} 1 - \cos t\sin t\, dt = t - \frac{\sin^2 t}{2}\Big|_{t=0}^{t=2\pi} = \boxed{2\pi}$$

$$\text{Flux}(F) = \oint_C F\cdot\vec{n}\,ds = \int_0^{2\pi} \langle \cos t, \cos t, -\sin t\rangle \cdot \frac{\langle \cos t, 2\sin t, \cos t\rangle}{\sqrt{2\cos^2 t + 4\sin^2 t}} \sqrt{1+\sin^2 t}\,dt$$

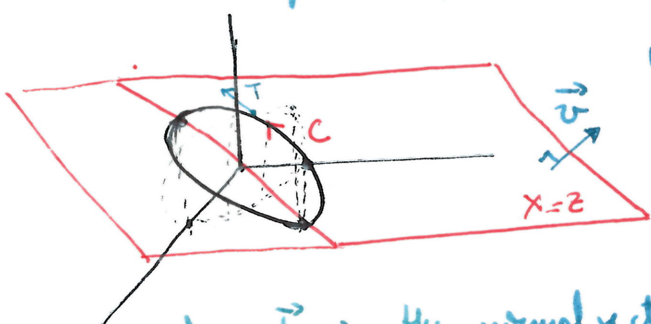
$$= \int_0^{2\pi} (\cos^2 t + 2\sin t\cos t - \cos t\sin t)\frac{\sqrt{1+\sin^2 t}}{\sqrt{2}\sqrt{1+\sin^2 t}}\,dt$$

$$= \frac{1}{\sqrt{2}}\int_0^{2\pi} \cos^2 t + \cos t\sin t\,dt = \frac{1}{\sqrt{2}}\int_0^{2\pi} \frac{1+\cos 2t}{2} + \cos t\sin t\,dt$$

$$\begin{cases} \cos^2 t + \sin^2 t = 1 \\ \cos^2 t - \sin^2 t = \cos 2t \end{cases} \Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{t}{2} + \frac{\sin 2t}{4} + \frac{\sin^2 t}{2} \Big|_{t=0}^{t=2\pi} \right) = \frac{2\pi}{2\sqrt{2}} = \boxed{\frac{\pi}{\sqrt{2}}}$$

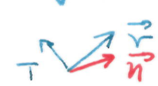
We can compute the flux in an alternative way, using the geometry of the curve



Notice $x = z$ contains the ellipse

$x=z$ has normal vector $\langle -1, 0, +1\rangle$

If the curve were in the xy-plane, then $\vec{n} = \vec{T}\times\vec{k}$



where $\vec{k}$ is the normal vector to the xy-plane.

<u>Conclusion</u>: we can take $\vec{n} = -\vec{T}\times\vec{v} = \vec{v}\times\vec{T}$ where $\vec{v}$ is normal to the plane containing the curve.

$$\vec{T} = \frac{\langle -\sin t, \cos t, -\sin t\rangle}{\sqrt{1+\sin^2 t}} \Rightarrow \vec{n} = \frac{-1}{\sqrt{1+\sin^2 t}}\begin{vmatrix} i & j & k \\ -\sin t & \cos t & -\sin t \\ 1 & 0 & -1 \end{vmatrix}$$

$$= \frac{-1}{\sqrt{1+\sin^2 t}}\langle -\cos t, -2\sin t, -\cos t\rangle$$

So $\text{Flux}(C) = \frac{\pi}{\sqrt{2}}$ as computed above.

<u>Notice</u>: If we had chosen $\langle 1, 0, -1\rangle$ as the normal vector, then $\vec{n} = \vec{T}\times\vec{v}$ would give the outer normal $\Rightarrow$ we need to be careful when choosing such vectors to get the orientation right.