MATH 2153 - Calculus III – Recitation 12

Prof. Cueto - The Ohio State University

April 21, 2016

1. **Horizontal channel flow.** Consider the velocity vector field $\mathbf{v}_{(x,y,z)} = \langle 0, 1 - x^2, 0\rangle$ for $|x|, |z| < 1$, which represents a horizontal flow in the y-direction. Suppose you place two paddle wheels at the points $P = (\frac{1}{2}, 0, 0)$ and $Q = (-\frac{1}{2}, 0, 0)$, respectively.

   (a) In which of the coordinate directions should the axes of the wheels point in order for the wheel to spin? In which directions do they spin?

   (b) Compute the graph of the curl of $\mathbf{v}$ and provide an interpretation of your findings in (a).

2. Evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \bullet \vec{n}\, dS$, where $\mathbf{F}_{(x,y,z)} = \langle -xz, yz, xye^z\rangle$, the region S in $\mathbb{R}^3$ is the cap of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 3$ and $\vec{n}$ is the unit normal vector to S pointing upwards. (*Hint:* Draw a picture of S.)

3. **Flux across a sphere.** Consider the radial vector field $\mathbf{F} = \langle x, y, z\rangle$ and let S be the spehere of radius a centered at the origin. Compute the outward flux of $\mathbf{F}$ across S. (*Hint:* Use the symmetry of the sphere and parameterize either one of the hemispheres)

4. Find the net outward flux of a general rotational vector field $\mathbf{F}_{(x,y,z)} = \vec{a} \times \langle x, y, z\rangle$ across any smooth closed surface, where $\vec{a} \in \mathbb{R}^3$ is a fixed non-zero vector.

5. **Compound surface and boundary.** Begin with the paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with the plane $y = 0$. Let S be the surface that remains for $y \ge 0$, including the planar surface in the xz-plane). Let C be the semicircle and line segment that bounds the cap of S in the plane $z = 4$, with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y, 2x + z, 2y + x \rangle$.

   (a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

   (b) Evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \vec{n}\, dS$, for $\vec{n}$ the unit outward normal to S from (a).

   (c) Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$ using (b).

   (d) Can you find another surface bounded by C? Would that surface simplify the calculations in (c)?

6. **Tilted disks** Let S be the disk enclosed by the curve C with parameterization $\mathbf{r}(t) = \langle \cos\varphi \cos t, \sin t, \sin\varphi \cos t \rangle$ for $0 \le t \le 2\pi$, where φ is a *fixed angle*.

   (a) Compute the area of S and the length of C.

   (b) What is the circulation of C of the vector field $\mathbf{F} = \langle -y, x, 0 \rangle$ as a function of the angle φ? What values of φ maximize the circulation?

   (c) What is the circulation of C of the vector field $\mathbf{F} = \langle -y, -z, x \rangle$ as a function of the angle φ? What values of φ maximize the circulation?