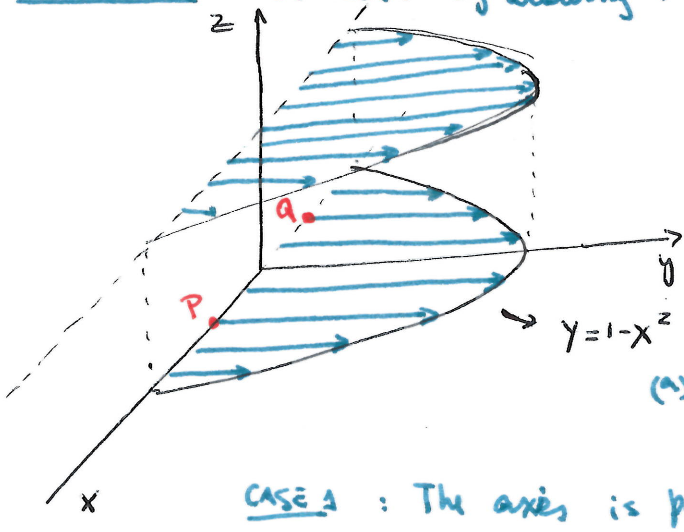
Recitation XII

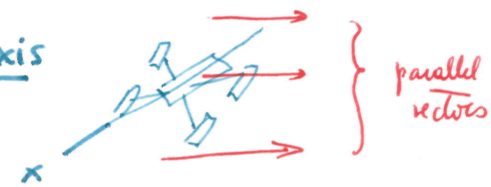
Problem 1: We start by drawing the velocity vector field along the x-axis.



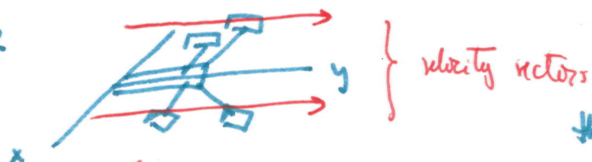
- The position vectors are placed along the parabola $y=1-x^2$ with their heads.
- They run in planes parallel to the xy-plane, since $\vec{r}$ is independent of the value of z.

(a) We have 3 possible positions for the axes of the paddle wheels.

CASE 1 : The axes is parallel to the x-axis so the paddles won't spin.

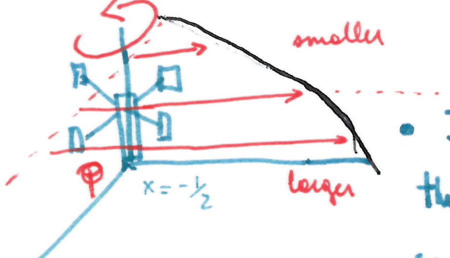


CASE 2

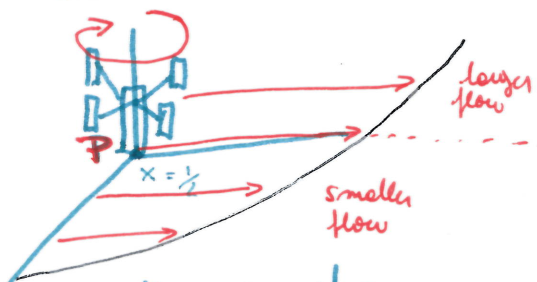


If the axis is parallel to the y-axis, then the paddles are parallel to the velocity, so also no spin!

CASE 3



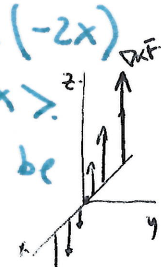
- If the axis at Q is parallel to the z-axis, the flow for $x<-\frac{1}{2}$ is smaller than for $x>-\frac{1}{2}$, so the wheel will spin COUNTERCLOCKWISE!
- If the axis at P is parallel to the z-axis, the flow for $x<\frac{1}{2}$ is greater than for $x>\frac{1}{2}$, so the wheel spins CLOCKWISE



Note: For any other direction $\vec{a}=\langle a_1, a_2, a_3\rangle$, the spin will be determine by the sign of a_3..

(b) We compute the curl $(\vec{r}) = \nabla\times\vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 1-x^2 & 0 \end{vmatrix} = i\,0 - j\,0 + k(-2x)$
$= \langle 0, 0, -2x\rangle$.

the curl has z-axis direction. This says the spin of the paddle will be maximal if the axis of the paddle is ligned up w/ the z-axis!

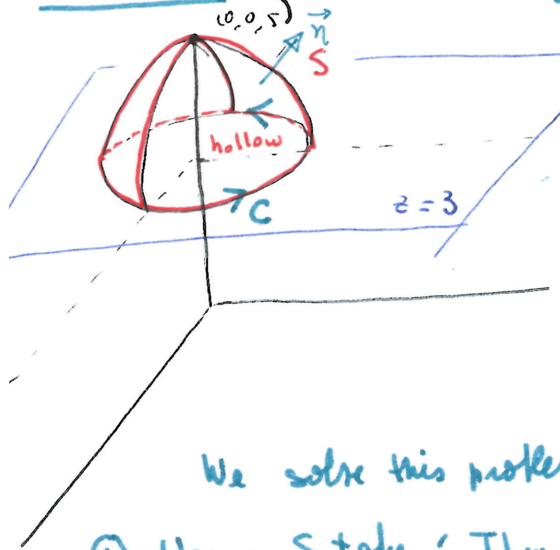


Note: $\vec{n} = \langle 0,0,1\rangle$ is normal to xy-plane. The flow at a point is determined by the value $(\nabla\times\vec{r})\cdot\vec{n} = -2x$

$(\nabla\times\vec{r})\cdot\vec{n} = -2x > 0$ at Q $\Rightarrow$ counterclock. rotation

$(\nabla\times\vec{r})\cdot\vec{n} = -2x < 0$ at P $\Rightarrow$ clockwise rotation

Problem 2: We start by drawing the surface.



We see that the surface is bounded by a circle C in the $z=3$ plane:

$$3 = 5 - x^2 - y^2 \iff x^2+y^2 = (\sqrt{2})^2 \quad \text{center} = (0,0), \ \text{radius} = \sqrt{2}$$

The normal vector is pointing upwards, so the bounding curve C must be oriented counterclockwise.

We solve this problem in various ways:

① Using Stokes' Thm! The curve is $\vec{r}(t) = \langle \underset{x}{\sqrt{2}\cos t}, \underset{y}{\sqrt{2}\sin t}, \underset{z}{3} \rangle$ $0 \le t \le 2\pi$ is counterclock. oriented as we want.

$$\iint_S (\nabla \times \vec{F})\cdot \vec{n}\, dS \stackrel{\downarrow}{=} \oint_C \vec{F}\cdot dr = \int_0^{2\pi} \langle (-\sqrt{2}\cos t)\cdot 3, (\sqrt{2}\sin t)3, 2\sin t\cos t e^3 \rangle \cdot \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\rangle\, dt$$

$r'(t) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\rangle$

$$= \int_0^{2\pi} \underbrace{6\cos t\sin t + 6\cos t \sin t + 0}_{=12\cos t\sin t}\, dt = 6\sin^2 t \Big|_{t=0}^{t=2\pi} = \boxed{0.}$$

Notice The vector field is not conservative since $f_z = -x \neq h_x = ye^z$ on S. ($\vec{F} = \langle f, g, h\rangle$)

② We add the disk bounded by C on $z=3$. The normal is $\langle 0,0,1\rangle$ (direction is compatible w/ the orientation of C).

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \oint_C \vec{F}\cdot dr = \iint_{\text{disc}} (\nabla\times\vec{F})\cdot\vec{n}\, dS.$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xz & yz & xye^z \end{vmatrix} = \langle xe^z - y, -(ye^z + x), 0\rangle \quad \text{so}$$

$$(\vec{\nabla}\times\vec{F})\cdot \underset{\langle 0,0,1\rangle}{\vec{n}} = 0 \quad \& \text{ so}$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_{\text{disc}} 0\, dS = \boxed{0} \ ✓$$

③ The integral can be computed from the definition of double. For this we first need to parameterize the surface as the graph of a function over the disk of radius $\sqrt{2}$.

$$\vec r(x,y) : \{x^2+y^2 \le 2\} \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (x, y, \overbrace{5-x^2-y^2}^{=z=P(x,y)})$$

$$\vec n \overset{?}{=} \pm \frac{\left(\frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y}\right)}{\left|\frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y}\right|} \qquad \frac{\partial \vec r}{\partial x}\times\frac{\partial \vec r}{\partial y} = \langle -P_x, -P_y, 1\rangle = \langle 2x, 2y, 1\rangle$$

points upwards so the sign should be $+$.

So
$$\iint_S (\nabla\times\vec F)\cdot\vec n\, dS = \iint_{x^2+y^2\le 2} \langle x e^{5-x^2-y^2} - y, -x - y e^{5-x^2-y^2}, 0\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$$

use $\nabla\times\vec F$ from ② & evaluate

$$= \iint_{x^2+y^2\le 2} x^2 e^{5-x^2-y^2} - 2xy - 2xy - y^2 e^{5-x^2-y^2}\, dA = \iint_{x^2+y^2\le 2} (x^2-y^2) e^{5-x^2-y^2} - 4xy\, dA$$

Use polar coordinates! $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$ $\quad 0 \le \rho \le 2$, $0 \le \theta \le 2\pi$.

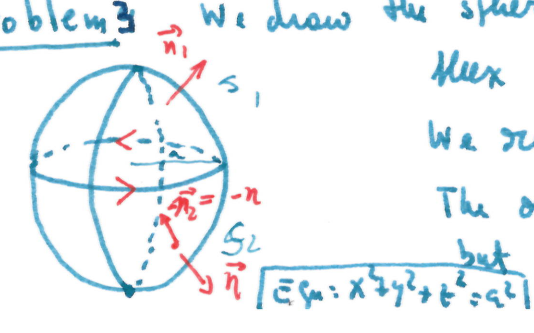
$$= \int_0^2\int_0^{2\pi} \Big(\rho^2 \underbrace{(\cos^2\theta - \mathrm{sen}^2\theta)}_{=\cos 2\theta} e^{5-\rho^2} - 4\rho^2 \underbrace{(\cos\theta\,\mathrm{sen}\,\theta)}_{=\frac{\mathrm{sen}\,2\theta}{2}}\Big)\rho\, d\theta\, d\rho$$

$$= \int_0^2\int_0^{2\pi} \rho^3 e^{5-\rho^2}(\cos 2\theta) - 2\rho^3\,\mathrm{sen}\,2\theta\, d\theta\, d\rho$$

$$= \int_0^2 \left(\rho^3 e^{5-\rho^2}\frac{\mathrm{sen}\,2\theta}{2}\Big|_{\theta=0}^{\theta=2\pi} + 2\rho^3\frac{\cos 2\theta}{2}\Big|_{\theta=0}^{\theta=2\pi}\right) d\rho = \int_0^2 0 + 0\, d\rho = \boxed{0}$$

Notice: Integrating in the order $\int_0^{2\pi}\int_0^2 \_\, d\rho\, d\theta$ would have been much more challenging!

Problem 3 We draw the sphere & use the equator to divide the calculation into the flux along 2 hemispheres.



We orient the curve counterclockwise from above.

The outer normals of S & S_1 agree ($\vec n = \vec n_1$) but the $\vec n = -\vec n_2$ on S_2 (because of the orientation of the curve).

• $\iint_{S_1} \vec{F}\cdot\vec{\eta}_1\, dS = ?$ We parameterise the northern hemisphere as

$$\vec{r}_1(x,y) : \{x^2+y^2\le a^2\} \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (x,y,\sqrt{a^2-x^2-y^2})$$

$$\frac{\partial \vec{r}_1}{\partial x}\times\frac{\partial \vec{r}_1}{\partial y} = \langle -f_x, -f_y, 1\rangle = \langle \frac{-x}{\sqrt{\ }}, \frac{-y}{\sqrt{\ }}\rangle$$ upwards ⇒ same direction as $\vec{\eta}_1$. $f(x,y)$

$$\iint_{S_1} \vec{F}\cdot\eta_1\, dS = \iint_{x^2+y^2\le a^2} \langle x, y, \sqrt{a^2-x^2-y^2}\rangle \cdot \langle \frac{+x}{\sqrt{a^2-x^2-y^2}}, \frac{+y}{\sqrt{a^2-x^2-y^2}}, 1\rangle dA$$

$$= \iint_{x^2+y^2\le a^2} \frac{+x^2+y^2+(a^2-x^2-y^2)}{\sqrt{a^2-x^2-y^2}}\, dA = \iint_{x^2+y^2\le a^2} \frac{a^2}{\sqrt{a^2-x^2-y^2}}\, dA \qquad (*)$$

• For S_2, we use the parameterization $\vec{r}_2(x,y) : \{x^2+y^2\le a^2\} \longrightarrow \mathbb{R}^3$
$$(x,y) \longmapsto (x,y,-\sqrt{a^2-x^2-y^2})$$

$$\frac{\partial \vec{r}_2}{\partial x}\times\frac{\partial \vec{r}_2}{\partial y} = \langle -f_x, -f_y, 1\rangle = -\vec{\eta}_2$$ upwards → we should reverse the sign of the integral
not same dir. as $\vec{\eta}_2$!

$$\iint_{S_2} \vec{F}\cdot\vec{\eta}\, dS = -\iint_{x^2+y^2\le a^2} \langle x, y, -\sqrt{a^2-x^2-y^2}\rangle \cdot \langle \frac{-x}{\sqrt{a^2-x^2-y^2}}, \frac{-y}{\sqrt{a^2-x^2-y^2}}, 1\rangle dA$$

$$= -\iint_{x^2+y^2\le a^2} \frac{-x^2-y^2-(a^2-x^2-y^2)}{\sqrt{a^2-x^2-y^2}}\, dA = \iint_{x^2+y^2\le a^2} \frac{a^2}{\sqrt{a^2-x^2-y^2}}\, dA$$ same as (*)

To solve (*) we use polar coordinates
$x=\rho\cos\theta$, $y=\rho\sin\theta$, $0\le\rho\le a$, $0\le\theta\le 2\pi$

$$(*) = \int_0^a\int_0^{2\pi} \frac{a^2}{\sqrt{a^2-\rho^2}}\,\rho\, d\theta\, d\rho = \int_0^{2\pi}\int_0^a \frac{a^2\rho}{\sqrt{a^2-\rho^2}}\, d\rho\, d\theta = \int_0^{2\pi} -a^2\sqrt{a^2-\rho^2}\Big|_{\rho=0}^{\rho=a} d\theta =$$

$$\frac{\rho}{\sqrt{a^2-\rho^2}} = \frac{d}{d\rho}\left(-\sqrt{a^2-\rho^2}\right)$$

$$= \int_0^{2\pi} \underbrace{a^2\sqrt{a^2}}_{=a^3}\, d\theta = \boxed{2\pi a^3}$$

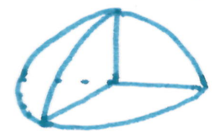
Conclusion: Outward Flux $= \boxed{4\pi a^3}$

Easier: Use Divergence Theorem! $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot F\,dV$

$$\nabla\cdot F = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3.$$

$$\iiint_D \nabla F\,dV = 3\iiint_D dV \equiv 3\Big(\underbrace{\iiint dV}_{\text{Top Half}} + \underbrace{\iiint dV}_{\text{Both Half}}\Big) = 6\underbrace{\iiint 1\,dV}_{\text{Top Half}}$$

Parametering the top half using spherical coordinates



$0 \le \rho \le a$
$0 \le \theta \le 2\pi$
$0 \le \varphi \le \frac{\pi}{2}$

$$\text{Vol (Top Half)} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho^2 \operatorname{sen}\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \rho^3\frac{\operatorname{sen}\varphi}{3}\Big|_0^a d\varphi\,d\theta = \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi/2} \operatorname{sen}\varphi\,d\varphi\,d\theta = \frac{-a^3}{3}\int_0^{2\pi} \cos\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}} d\varphi$$

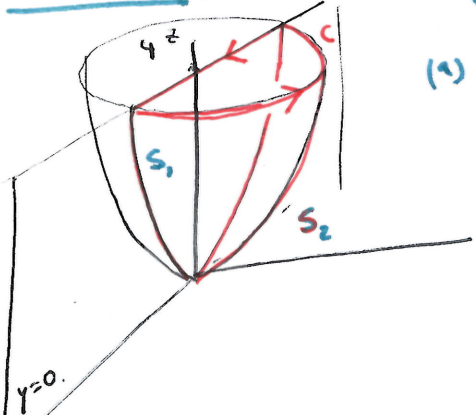
$$= \frac{a^3}{3}\int_0^{2\pi} d\varphi = \boxed{\frac{2\pi a^3}{3}}$$

Conclusion: $\iint_S F\cdot\vec{n} = 6\cdot\frac{2\pi a^3}{3} = 4\pi a^3$ as we did before!

Problem 4: We computed $\operatorname{div}(\vec{F}) = \operatorname{div}(\vec{a}\times\langle x,y,z\rangle) = 0$ (Lecture XXXV)

Using the Divergence Thm: $\iint_S F\cdot\vec{n}\,dS = \iiint_D 0\,dV = 0.$

Problem 5: We start by drawing the picture

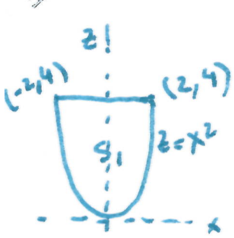


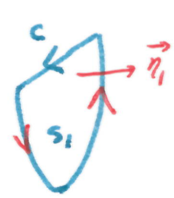
(a) We use the right hand rule to decide the direction of the normal vectors.

Write $S = S_1 \cup S_2$, where

$$S_1 = \{\, y=0,\ 4 \ge z \ge x^2+y^2 \,\} \quad \text{"}x^2\text{"}$$

$$S_2 = \{\, z = x^2+y^2 \quad y \ge 0 \,\} \quad 0 \le z \le 4.$$





$\vec{n}_1 = \langle 0,1,0\rangle$ because the line segment on C is traversed clockwise from behind. We can extend the orientation to an orientation of the parabola $z=x^2$ on the xz-plane.

Similarly, we use the semicircle and extend to an orientation of the curve bounding S_2  so the normal vector is pointing towards the xz-plane. (last component of $\vec{n}_2$ is positive & y-comp. is negative)

We find the expression for $\vec{n}_2$ from a parameterization of S_2:

$\vec{r}_2 : R_2 \to \mathbb{R}^2$, $\vec{r}_2(x,y) = \langle x, y, \underbrace{x^2+y^2}_{f(x,y)} \rangle$. where $R_2 = \{ y \geq 0 \quad x^2+y^2 \leq 4 \}$

$$\frac{\partial \vec{r}_2}{\partial x} \times \frac{\partial \vec{r}_2}{\partial y} = \langle -f_x, -f_y, 1 \rangle \Rightarrow \begin{array}{l} f_x = 2x \\ f_y = 2y > 0 \end{array} \quad \text{so } \vec{n}_2 = \frac{\frac{\partial \vec{r}_2}{\partial x} \times \frac{\partial \vec{r}_2}{\partial y}}{\left| \frac{\partial \vec{r}_2}{\partial x} \times \frac{\partial \vec{r}_2}{\partial y} \right|}$$

$$\vec{n}_2 = \frac{\langle -2x, -2y, 1 \rangle}{\sqrt{1+4x^2+4y^2}}$$. point to the inside of the paraboloid.

(b) We start by computing $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+y & 2x+z & 2y+x \end{vmatrix} = \langle 2-1, -(1-2), 2-1 \rangle = \langle 1,1,1 \rangle$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, dS + \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$$\iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{\substack{x^2+y^2 \leq 4 \\ y \geq 0}} \langle 1,1,1 \rangle \cdot \langle -2x, -2y, 1 \rangle \, dA = \iint_{\substack{x^2+y^2 \leq 4 \\ y \geq 0}} -2x-2y+1 \, dA$$

Use polar coordinates $\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$  $0 \leq \rho \leq 2$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2\rho\cos\theta - 2\rho\sin\theta + 1) \rho \, d\theta \, d\rho = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 -2\rho^2(\cos\theta + \sin\theta) + \rho \, d\rho \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -\frac{2}{3}\rho^3(\cos\theta+\sin\theta) + \frac{\rho^2}{2} \Bigg|_{\rho=0}^{\rho=2} d\theta = \int_{-\pi/2}^{\pi/2} -\frac{16}{3}(\cos\theta + \sin\theta) + 2 \, d\theta$$

$$= -\frac{16}{3}(\sin\theta - \cos\theta) \Bigg|_{\theta=\pi/2}^{\theta=\pi/2} + 2\pi = 2\pi - \frac{16}{3}\left(((1-0)-(-1-0)) \right) = \boxed{2\pi - \frac{32}{3}}$$

. S_1 is a flat region in the xz-plane so $dS = dx\,dz = dA_{xz}$.

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_{S_1} \langle 1,1,1 \rangle \cdot \langle 0,1,0 \rangle \, dA = \text{Area}(S_1) = \int_{-2}^{2} \int_{x^2}^{4} 1 \, dz \, dx = \int_{-2}^{2} 4-x^2 \, dx$$

$\boxed{S_1 : \begin{array}{l} -2 \leq x \leq 2 \\ x^2 \leq z \leq 0 \end{array}}$

$$= 4x - \frac{x^3}{3} \Bigg|_{-2}^{2} = \frac{32}{3}$$

Conclusion : $\displaystyle\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = 2\pi - \frac{32}{3} + \frac{32}{3} = \boxed{2\pi}$

(c) Using Stokes' Theorem $\displaystyle\oint_C F\cdot dr = \iint_S (\nabla\times\vec{F})\cdot\vec{n} = 2\pi.$

(d) C also bounds the surface  on the plane $z=4$. whose normal is $\vec{n}_3 = k$.

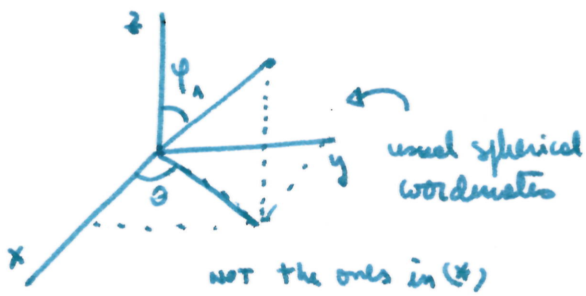
By Stokes $\displaystyle\iint_S (\nabla\times\vec{F})\cdot\vec{n} = \iint_{S_3} (\nabla\times\vec{F})\cdot\vec{n}_3\, dS_3 = \oint_C \vec{F}\cdot dr$.

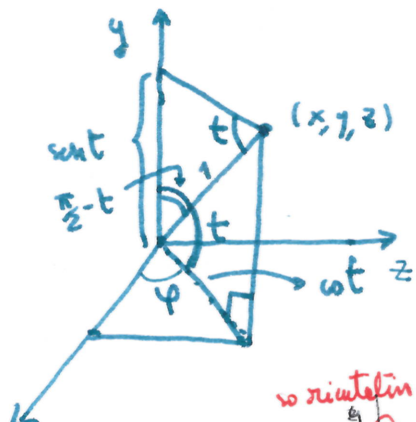
S_3 :  $0\le\rho\le 2$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ so $\displaystyle\iint_{S_3} (\nabla\times\vec{F})\cdot\underset{\langle 0,0,1\rangle}{\underset{\shortparallel}{\vec{n}_3}}\, dS_3 = \iint_{S_3} 1\, dS_3 = \text{Area}\left(\tfrac{1}{2}\text{ circle}\right) = \frac{\pi\cdot 2^2}{2} = \boxed{2\pi}$

Problem 6: We use the angle φ as the inclination of a plane containing the circle C enclosing S.

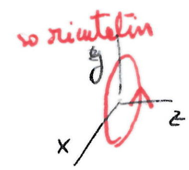
Notice : $x^2+y^2+z^2 = \underline{\cos^2\varphi}\cos^2 t + \sin^2 t + \underline{\sin^2\varphi}\cos^2 t = \cos^2 t + \sin^2 t = 1$ so the circle lies in the unit sphere.

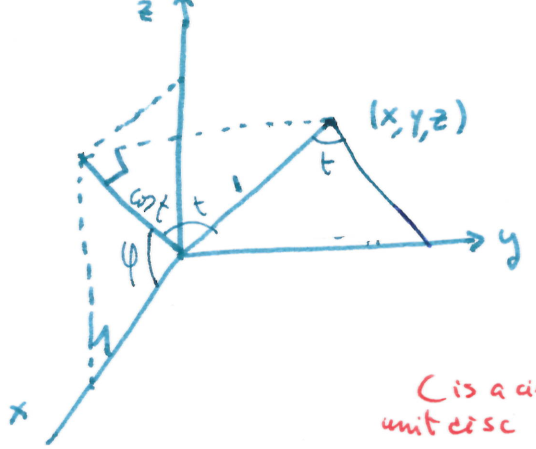
$$(*)\begin{cases} x = \cos\varphi\cos t \\ y = \sin t \\ z = \sin\varphi\cos t \end{cases}$$



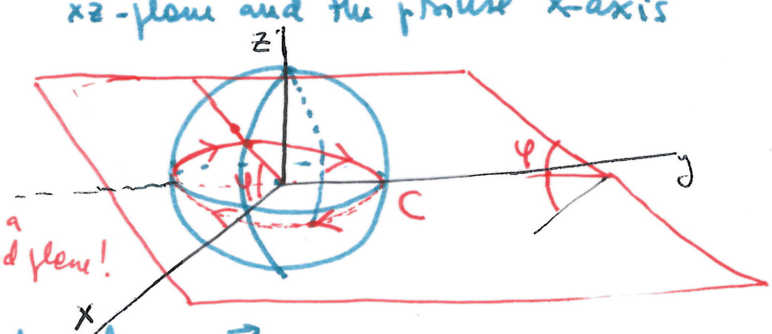


So instead of projecting to the xy-plane, we project to the xz-plane:





φ is the angle between the projection to the xz-plane and the positive x-axis



C is a circle bounding a unit disc on an inclined plane!

(a) We compute the area as a surface integral. $\vec{r}(t,\rho) = \langle \rho\cos\varphi\cos t, \rho\sin t, \rho\sin\varphi\cos t\rangle$
$0\le\rho\le 1 \quad 0\le t\le 2\pi.$

$\displaystyle\text{Area}(S) = \iint_S 1\, dS$

$\vec{r}_t = \langle -\rho\cos\varphi \sin t, \rho\cos t, -\rho\sin\varphi\sin t\rangle$

$\vec{r}_\rho = \langle \cos\varphi\cos t, \sin t, \sin\varphi\cos t\rangle$

$$\vec{r}_t \times \vec{r}_\rho = \begin{vmatrix} i & j & k \\ -\rho\cos\varphi\sin t & \rho\cos t & -\rho\sin\varphi\sin t \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \end{vmatrix} = \langle \rho\sin\varphi\cos^2 t + \rho\sin\varphi\sin^2 t, -(-\rho\cos\varphi\sin\varphi\cos t\sin t + \rho\sin\varphi\sin t\cdot\cos\varphi\cos t), -\rho\cos\varphi(\sin^2 t+\cos^2 t)$$

$$= \langle \rho\sin\varphi, 0, -\rho\cos\varphi \rangle$$

$$\Rightarrow |\vec{r}_t\times\vec{r}_\rho| = \sqrt{\rho^2\sin^2\varphi + \rho^2\cos^2\varphi} = |\rho| = \rho > 0 \checkmark$$

So Area $= \int_0^1\int_0^{2\pi} \rho \, dt\, d\rho = 2\pi \left.\frac{\rho^2}{2}\right|_0^1 = \boxed{\pi}$ (It's a circle of radius 1 but in a plane $\neq$ xy-plane)

- length of $C = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$ = length of a circle of radius 1 but in a plane $\neq$ xy-plane

$\vec{r}(t) = \langle \cos\varphi\cos t, \sin t, \sin\varphi\cos t\rangle$

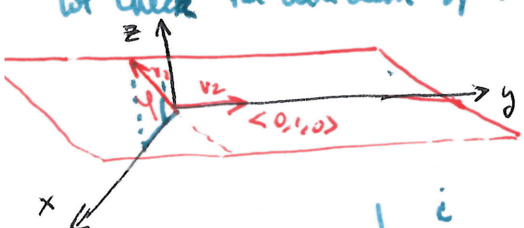
$\vec{r}\,'(t) = \langle -\cos\varphi\sin t, \cos t, \sin\varphi\sin t\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{\cos^2\varphi\sin^2 t + \cos^2 t + \sin^2\varphi\sin^2 t}$

$= \sqrt{\sin^2 t + \cos^2 t} = 1$

(b) $\oint_C F\cdot dr = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$. Note: C is oriented counterclockwise from the xy-plane

Compute $\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \langle 0, -0, 1-(-1)\rangle = \langle 0,0,2\rangle$

What's $\vec{n}$? We find 2 directions in the plane and compute their cross product. Then, we check the direction of $\vec{n}$ that is compatible with the orientation of C.



$$\begin{cases} v_2 = \langle 0,1,0\rangle \\ v_1 = \langle \cos\varphi, 0, \sin\varphi\rangle \end{cases}$$

$\Rightarrow v_1\times v_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ \cos\varphi & 0 & \sin\varphi \end{vmatrix} = \langle \sin\varphi, 0, -\cos\varphi\rangle$ is pointing to the neg X-axis (and is not compatible w/ the orientation of C.)

Take $\vec{n} = v_2\times v_1 = \langle -\sin\varphi, 0, \cos\varphi\rangle$. parameters for S are t & ρ. $\begin{cases} 0\le t\le 2\pi \\ 0\le\rho\le 1\end{cases}$

$$\Rightarrow \oint_C F\, dr = \iint_S \langle 0,0,2\rangle\cdot\langle -\sin\varphi, 0, \cos\varphi\rangle\, dS = \int_0^1\int_0^{2\pi} 2\cos\varphi\,\rho\, dt\, d\rho$$

$$= \int_0^1\int_0^{2\pi} \cos\varphi\, 2\rho\, dt\, d\rho = \int_0^1 2\rho\left(\int_0^{2\pi}\cos\varphi\, dt\right)d\rho = \int_0^1 2\rho\cos\varphi\, 2\pi\, d\rho = 2\pi\cos\varphi \left.\rho^2\right|_0^1 = \boxed{2\pi\cos\varphi}$$

$\Rightarrow$ maximal when $\varphi = 0$ (plane is the xy-plane)

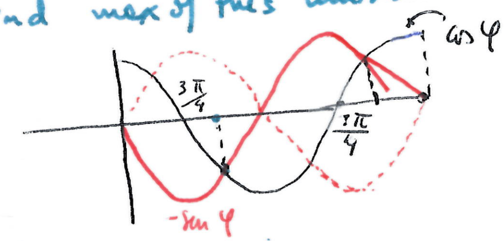
c) We proceed in the same fashion for the new $\vec{F}$ $\qquad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -z & x \end{vmatrix} = \langle 1,-1,1 \rangle$

$$\oint_C \vec{F}\, dr = \iint_S \langle 1,-1,1\rangle \langle -\sin\varphi, 0, \cos\varphi \rangle \, dS = \int_0^1 \int_0^{2\pi} (\cos\varphi - \sin\varphi)\rho \, dt \, d\rho$$

$$= (\cos\varphi - \sin\varphi)\ 2\pi \int_0^1 \rho \, d\rho = (\cos\varphi - \sin\varphi)\, 2\pi \frac{\rho^2}{2}\Big|_0^1 = \boxed{\pi(\cos\varphi - \sin\varphi)}$$

Circulation is maximal when $\cos\varphi - \sin\varphi$ is maximal.

$h(\varphi) = \cos\varphi - \sin\varphi \qquad (0 \le \varphi \le \pi.)$ $\longrightarrow$ Find max of this univariate function



$h'(\varphi) = -\sin\varphi - \cos\varphi = 0 \iff \cos\varphi = -\sin\varphi$

$\varphi = \frac{3\pi}{4}$

$h(0) = h(2\pi) = 1 - 0 = 1$

$h\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$
$= -\sqrt{2}$

Max circulation: when $\varphi = 0$.