Review Final (§14.6–15.8)

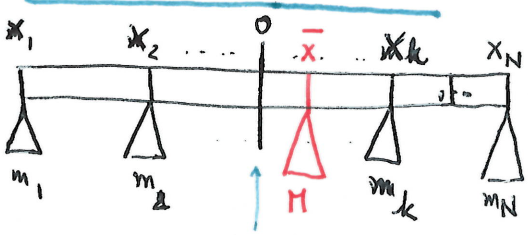
§0 Topics from midterm 1 & 2 covered before (see Notes on course's website)

§ 14.6 Integrals for mass calculations:

Def: The center of mass or centroid of an object is the point where all of the mass of the object is concentrated. It's the point all of the mass of the object would be located if it were treated as a point mass.

4 cases:

(A) Discrete case in 1-dim:



$M = \sum_{k=1}^{N} m_k$ total mass

N Objects: each of mass m_k & located at the point x_k in $\mathbb{R}$

Center of mass (location): $\bar{x} = \dfrac{\sum_{k=1}^{N} m_k \boxed{x_k}}{\sum_{k=1}^{N} m_k}$ in $\mathbb{R}$

$\bar{x}$: $\boxed{\text{balance point}}$

weigh mass of object by signed distance to the origin.

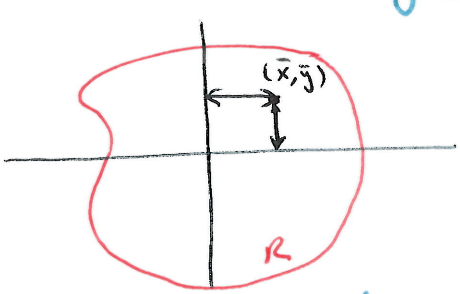
(B) Continuous case in 1-dim: replace sum by integration.

- rod, represented by an interval $[a,b]$ in $\mathbb{R}$
- discrete mass: replaced by a density function $\rho(x)$

$\Rightarrow$ Total mass of the rod $= \int_a^b \rho(x)\,dx$.

- balance point $\bar{x} = \dfrac{\int_a^b x\,\rho(x)\,dx}{\text{mass}}$ (Numerator = total momentum)

(C) 2-dimensional objects:



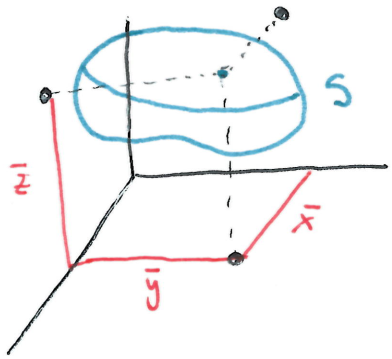
density function $\rho(x,y)$ $\rightsquigarrow$ $\text{Mass} = \iint_R \rho(x,y)\,dA$.

$\bar{x} = \dfrac{1}{\text{Mass}} \iint_R x\,\rho(x,y)\,dA$, $\bar{y} = \dfrac{1}{\text{Mass}} \iint_R y\,\rho(x,y)\,dA$.

Note. If ρ is constant, we can take it to be 1 to compute $\bar{x}$ & $\bar{y}$.

- Can use the geometry of R & symmetries of $\rho(x,y)$ to help our calculations (e.g. Quiz 4)

(D) 3-dimensional objects:



density $\rho(x,y,z) \rightsquigarrow$ mass $= \iiint_S \rho(x,y,z)\,dV$

$$\bar{x} = \frac{1}{\text{mass}} \iiint_S x\,\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{1}{\text{mass}} \iiint_S y\,\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{1}{\text{mass}} \iiint_S z\,\rho(x,y,z)\,dV$$

- Same observations from (C) apply here.

§14.7 Change of variables in multiple integrals:

Extend
- change of variables in $\mathbb{R}$
- cartesian to polar integration in $\mathbb{R}^2$
- " " spherical / cylindrical integration in $\mathbb{R}^3$

<u>Idea</u>: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(u,v) \longmapsto (g(u,v), h(u,v))$

<u>Need</u>: (*)
- T to be invertible (1-to-1) in the interior of S
- g, h must have continuous 1st partials in the interior of S.
- S & R (= image of S under the map T) are closed & bounded in $\mathbb{R}^2$

$\rightsquigarrow$ $\text{Jac}(u,v)$ (Jacobian) $= \begin{array}{c|cc|} & \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \\ g & g_u & g_v \\ h & h_u & h_v \end{array} = g_u h_v - g_v h_u$

Change of variables formula for a map $F(x,y)$ continuous:

<u>Theorem</u>: If 3 conditions in (*) hold, then:

$$\iint_R F(x,y)\,dA_{(x,y)} = \iint_S F(g(u,v), h(u,v))\,|\text{Jac}(u,v)|\,dA_{(u,v)}$$

↳ abs. value of Jac.

<u>Examples</u>: • polar to cartesian coords $\rightsquigarrow$ $\text{Jac} = \rho$
$u = \rho$, $v = \theta$

- Same idea works in $\mathbb{R}^3$: $T_{(u,v,w)} = (f, g, h)$ $\quad \text{Jac}_{(u,v,w)} = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$

<u>Thm</u>:
$$\iiint_R F(x,y,z)\,dV_{(x,y,z)} = \iiint_S F(f(u,v,w), g(u,v,w), h(u,v,w))\,|\text{Jac}_{(u,v,w)}|\,dV_{(u,v,w)}$$

Example: • cylindrical to cartesian $\rightsquigarrow$ $Jac = \rho$
(ρ, θ, z) (x, y, z)

• spherical to cartesian $\rightsquigarrow$ $Jac = \rho^2 \sin\varphi$.
(ρ, φ, θ) (x, y, z)

Strategies ① Aim for S to be a simpler region of integration than R.

② If the choice of transformation T is natural, getting $J_{(u,v)}$ is immediate but computing T is hard (need to invert T!)

Idea:

Find T^{-1} (inverse map)

• boundary of R $\rightleftarrows$ boundary of S
• Pick a point inside of R & see if it comes from int(S). If so, the whole int(S) maps to R

③ The function $f_{(x,y)}$ can suggest changes of coordinates (like it did in Calc I $\rightsquigarrow$ substitution methods!)

Eg: ⓐ $f_{(x,y)} = \sqrt{x-2y}\,(x-y)$ $\rightsquigarrow$ Pick $T^{-1}: \begin{cases} u = x-y & = f(x,y) \\ v = x-2y & = g(x,y) \end{cases}$ This gives T^{-1} so finding S is easy (it's the image of R under T^{-1}). To find $J_{(u,v)}$ we need to invert the map!

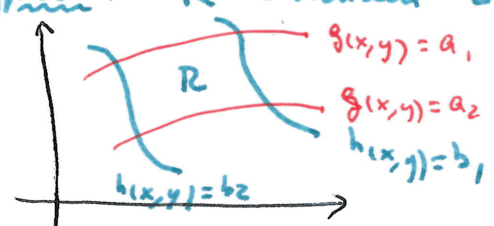
$$T: \begin{cases} x = 2u - v \\ y = u - v \end{cases} \Rightarrow J_{ac(u,v)} = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -1$$

ⓑ $f_{(x,y)} = (x+4y)^{3/2}$

$$\rightsquigarrow T^{-1}: \begin{cases} u = x+4y \\ v = y \end{cases} \rightsquigarrow T: \begin{cases} x = u - 4v \\ y = v \end{cases} \rightsquigarrow Jac_{(u,v)} = \begin{vmatrix} 1 & -4 \\ 0 & 1 \end{vmatrix} = 1.$$

④ The boundary of R can suggest changes of coordinates (eg Quiz 4)

Typical: R bounded by 2 pairs of parallel level curves



$$T^{-1}\begin{cases} u = g(x,y) \\ v = h(x,y) \end{cases} \rightsquigarrow S = \{(u,v) : \begin{array}{l} a_2 \le u \le a_1 \\ b_2 \le v \le b_1 \end{array}\}$$ is a rectangle

To find Jac, need to invert T^{-1}, ie. write $\begin{cases} x = x_{(u,v)} \\ y = y_{(u,v)} \end{cases}$ (as we did in Eg ⓐ & ⓑ above).

§15.1. Vector fields

• $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ v. field in $\mathbb{R}^2$, $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ v. field in $\mathbb{R}^3$

$\langle x,y \rangle \longmapsto \langle f_{(x,y)}, g_{(x,y)} \rangle$ $\quad$ $\langle x,y,z \rangle \longmapsto \langle f, g, h \rangle_{(x,y,z)}$

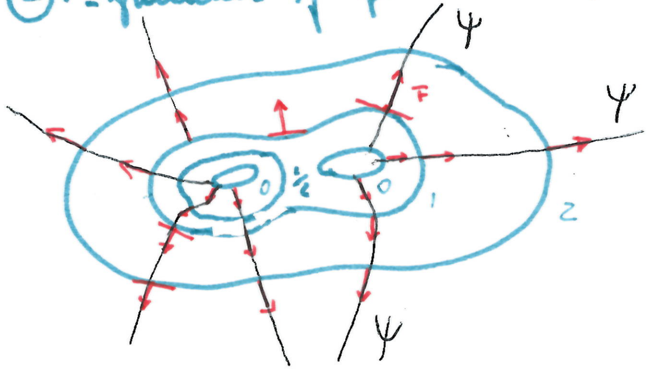
• F continuous / differentiable $\iff$ all its components are cont./differentiable

• Draw $F_{(x,y,z)}$ as a vector with tail at (x,y,z).

Examples: ① radial vector fields $F_{(x,y,z)} = \frac{\vec r}{|\vec r|^p} = \frac{\langle x,y,z\rangle}{|\langle x,y,z\rangle|^p}$ (p in $\mathbb{R}$)

If $p>0$, F is not defined on $(0,0,0)$.

② $\vec F$ = gradients of functions $\varphi: \mathbb{R}^2 \to \mathbb{R}$ / $\varphi: \mathbb{R}^3 \to \mathbb{R}$



• Contour map of φ = level curves of φ

• $\nabla\varphi$ is $\perp$ to level curves (equipotential curves)

• Can find curves Ψ perpendicular to the level curves of φ, so they are tangent to $\vec F$ (example in Rec. 11).

Lect 28

§15.2 Line integrals:

§1 Scalar line integrals

$f: \mathbb{R}^3 \to \mathbb{R}$ curve C in $\mathbb{R}^3$ parameterized by $\vec r_{(t)}: [a,b] \to \mathbb{R}^3$

$t \longmapsto \langle x_{(t)}, y_{(t)}, z_{(t)}\rangle$

Def $\int_C f_{(x,y,z)}\, ds = \int_a^b f(x_{(t)}, y_{(t)}, z_{(t)})\, |\vec r'_{(t)}|\, dt$.

ds → arc length

$|\vec r'_{(t)}| = \sqrt{x'^2_{(t)} + y'^2_{(t)} + z'^2_{(t)}}$

• Eg $f \equiv 1$, the integral gives the length of the curve.

• Typical C is described geometrically (eg lines or circles), and we need to find $\vec r_{(t)}$

• $\vec r_{(t)}$ induces an ORIENTATION on C

$\vec T$ = unit tangent $= \frac{\vec r'_{(t)}}{|\vec r'_{(t)}|}$

$\vec n = \vec n_{out}$ = unit "outward" normal = to the right of C as we walk along C

Note: At every point $\vec n = \pm N$ → from TNB frame

$\vec N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$ ⇝ directed to curving of C.

In $\mathbb{R}^2$: • $\vec n = \frac{1}{|r'_{(t)}|}\langle y'_{(t)}, -x'_{(t)}\rangle$

Eg: C = unit circle in $\mathbb{R}^2$:

• $\vec r_{(t)} = \langle \cos t, \sin t\rangle$ clockwise oriented

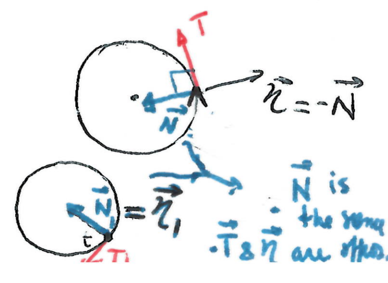
$\vec T = \langle -\sin t, \cos t\rangle$ ⇝ $\vec N = \langle -\cos t, -\sin t\rangle$ (from TNB-frame)

$\vec n = \langle \cos t, \sin t\rangle$

$\vec r_{1(t)} = \langle \cos t, -\sin t\rangle$

$\vec T_1 = \langle -\sin t, -\cos t\rangle$ ⇝ $\vec N_1 = \langle -\cos t, \sin t\rangle$

$\vec n_1 = \langle -\cos t, \sin t\rangle$



• $\vec N$ is the same. $\vec T$ & $\vec n$ are opp.

§2 Line integrals of v. fields:

$C: \vec r(t) = \langle x(t), y(t), z(t)\rangle$

$F = \langle f, g, h\rangle$

$$\mathrm{Circ}(\vec F, C) = \int_C \vec F\cdot\vec T\,ds = \int_C \vec F\cdot dr = \int_a^b \{ f(x(t),y(t),z(t))\cdot x'(t) + g(x(t),y(t),z(t))\cdot y'(t) + h(x(t),y(t),z(t))\cdot z'(t)\}\,dt.$$

$$\mathrm{Circ}(\vec F, \underbrace{C^{op}}_{\text{reverse orientation}}) = -\,\mathrm{Circ}(\vec F, C)$$

In $\mathbb{R}^2$: $\underbrace{\mathrm{Flux}(\vec F, C)}_{\text{flux across a curve.}} = \int_C \vec F\cdot n_{out}\,ds = \int_a^b f_{(x(t),y(t))}\, y'(t) - g_{(x(t),y(t))}\cdot x'(t)\,dt$

Application: Work of a force moving an object along a curve:

$$W = \int_C F\cdot\vec T\,ds.$$

§15.3 Conservative Vector fields

Def: F is conservative if $F = \nabla\varphi$ for some function φ (called potential)

Being conservative or not depends on the region where F is defined.

Tests ① $F = \langle f, g\rangle$ assume f, g have cont 1st partials.

If F is conservative, then $f_y = g_x$ $\left(\Longleftrightarrow \underbrace{\mathrm{curl}\,\vec F}_{g_x - f_y} = 0\right)$

② $F = \langle f, g, h\rangle$ with f, g, h have cont. 1st partials.

If F is conservative, then $f_y = g_x$, $f_z = h_x$ & $g_z = h_y$.

(Why? mixed partials of φ are cont, so they must agree) $\left(\Longleftrightarrow \mathrm{curl}\,\vec F = \nabla\times\vec F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \vec 0\right)$

Theorem: The test is sufficient if $\vec F$ is defined on an open, connected & simply connected region R.

any 2 pts can be joined by a curve inside R

no holes!

Method to find φ: $\langle f, g\rangle = \langle \varphi_x, \varphi_y\rangle \rightsquigarrow$ integrate f or g to find φ. (Rec 11 has examples).

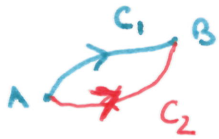
Path integrals & conservative v. fields:

Fund Theorem: $\displaystyle\int_C \nabla\varphi \cdot dr = \varphi(B) - \varphi(A)$ for

path independent! (integral only depends on the end points of C.

Theorem: If a vector field $\vec{F}$ has the path indep. property, then its conservative (no condition on the region R!)

Note: path independence holds (w.r.t any curve joining A & B) $\Longleftrightarrow$ $\displaystyle\oint_C F\cdot dr = 0$ for every simple closed curve C containing A & B.



Path independence property is stronger than conditions in the Test!

§15.4 Green's Theorem:

- R open, conn & simply connected region in $\mathbb{R}^2$ bounded by a curve C oriented COUNTERCLOCKWISE

- $F = \langle f, g\rangle$ cont v. field where f, g has cont 1st partials

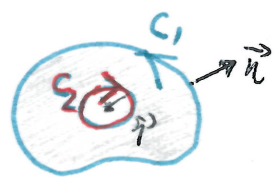
① Green's Thm (Circ form): $\displaystyle \mathrm{circ}(F, C) = \oint_C F\cdot dr = \iint_R \underset{g_x - f_y}{\mathrm{curl}\,\vec{F}}\, dA.$

② Green's Thm (Flux form) $\displaystyle \mathrm{Flux}_{out}(F, C) = \oint F\cdot \underset{\text{unit outward normal.}}{\vec{n}}\, dr = \iint_R \underset{f_x + g_y}{\mathrm{div}\,\vec{F}}\, dA$

Applications: Calculate area of R.

Take $\vec{F} = \langle 0, x\rangle$ or $\vec{F} = \langle y, 0\rangle$ & use Circ form.

For more general regions (with holes!)



C_1 & C_2 must have reverse orientation

$$\oint_{C_1} \vec{F}\cdot\vec{T}\, ds - \oint_{\substack{C_2\\ \text{(counterc)}}} \vec{F}\cdot\vec{T}\, ds = \iint_R \mathrm{curl}(\vec{F})\, dA$$

§15.5 Divergence & curl:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$\vec{F} = \langle f, g, h \rangle \rightsquigarrow \operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = f_x + g_y + h_z$

$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$ is a vector field!

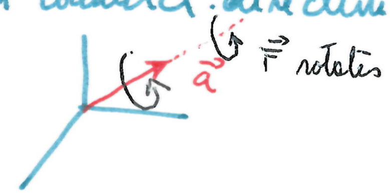
Each coord of curl indicates rotation direction

z-comp$(\operatorname{curl}(\vec{F})) > 0 \iff$ projection of $\vec{F}$ to xy-plane is rotating counterclockwise.

- Eg general rotational v. fields $\vec{F}_{(x,y,z)} = \vec{a} \times \vec{r}$ $\quad \vec{a} = \langle a_1, a_2, a_3 \rangle \neq \vec{0}$ fixed, $\vec{r} = \langle x, y, z \rangle$.

① $\operatorname{div} \vec{F} = 0$

② $\vec{F} = \vec{a} \times \vec{r}$ circles around the vector $\vec{a}$ in countercl. direction looking towards $\vec{0}$



③ $\nabla \times \vec{F} = 2\vec{a}$.

Thm: $\vec{F} = \langle f, g, h \rangle$ have cont 2^{nd} partials, then $\nabla \cdot (\nabla \times \vec{F}) = 0$.

§15.6 Surface integrals

Parametriz of a surface S in $\mathbb{R}^3$ $\quad \vec{r}_{(u,v)} = \langle x_{(u,v)}, y_{(u,v)}, z_{(u,v)} \rangle$.

Typical: domain R of $\vec{F} = \begin{cases} a \le u \le b \\ c \le v \le d \end{cases}$

Def: $$\iint_S f_{(x,y,z)} \, dS = \iint_R f(x_{(u,v)}, y_{(u,v)}, z_{(u,v)}) \left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| dA_{(u,v)}$$

Eg: S = graph of a function $\vec{r}_{(u,v)} = \langle u, v, P_{(u,v)} \rangle$. $\quad (S = z = P_{(u,v)})$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle -P_u, -P_v, 1 \rangle.$$

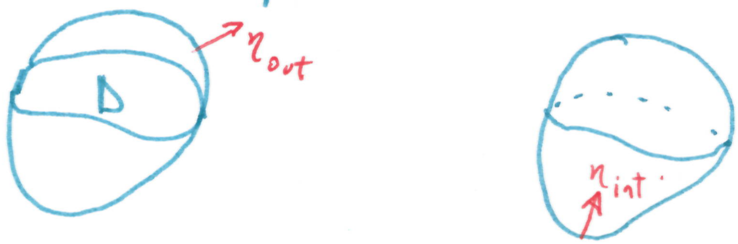
$$\rightsquigarrow \iint_S f_{(x,y,z)} \, dS = \iint_R f(u, v, P_{(u,v)}) \sqrt{1 + P_u^2 + P_v^2} \, dA.$$

§15.7 Stoke's Theorem:

- Orientation of the surface $S \longleftrightarrow$ continuous choice of a unit normal vector at each pt in S

- S closed (boundary of a region D in $\mathbb{R}^3$, open, conn. & simply connected)

  Then normal = points either inward or outwards



- S bounded by curve: orient the curve & use right hand rule.

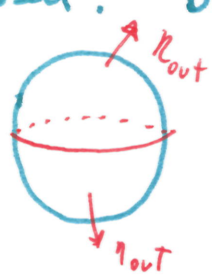
Recall: If $\vec{r}(u,v)$ parameterizes S: $\eta = \pm \dfrac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}$. choose the SAME SIGN through S

Stokes Thm: S bounded by a curve C & $\vec{\eta}_{out}$ compatible w/ orientation of C.

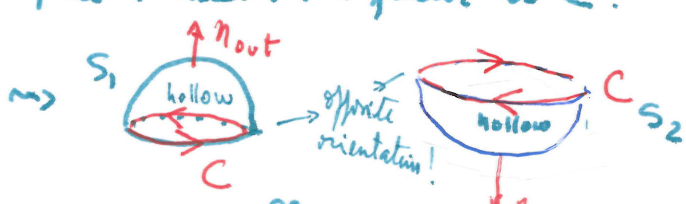
$$\operatorname{Circ}(F, C) = \oint_C F \cdot T\, ds = \iint_S \underbrace{(\nabla \times F)}_{\text{curl}} \cdot \vec{\eta}_{out}\, dS = \operatorname{Flux}(\nabla \times \vec{F}, S)$$

Q: What if S is closed? Break it in 2 pieces with a curve!

Example



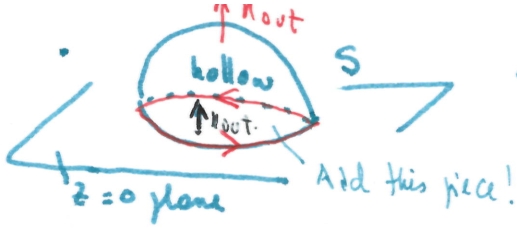
S sphere: uses the Equator as C.



$$\iint_S \nabla \times \vec{F}\ \eta_{out} = \iint_{S_1} \nabla \times \vec{F}\ \eta_{out} + \iint_{S_2} \nabla \times \vec{F} \cdot \eta_{out} \underset{\substack{\downarrow \\ \text{Stokes' on} \\ \text{each piece.}}}{=} \oint_C F \cdot T ds + \underbrace{\oint_C F \cdot T ds}_{= -\oint_C F \cdot T ds}$$

- Same is true for any smooth closed surface S.

so $= 0$



$$\int_C \vec F\cdot T\,ds \overset{\text{Stokes}}{\overset{!}{=}} \iint_S (\nabla\times\vec F)\cdot \eta_{out}\,dS$$

$\parallel$ Stokes

$$\iint_{\oslash} (\nabla\times\vec F)\cdot \eta_{out}\,dS \to \text{easier because it's a double integral in the plane!}$$

$\vec F = \langle f, g, h\rangle$

$\eta_{out} = \langle 0,0,1\rangle$.

so $(\nabla\times\vec F)\cdot\eta_{out}$ = z-comp of curl $= g_x - f_y$ ~~where~~ using Green's Thm!

§15.8 Divergence Theorem:

D open, conn & simply conn region in $\mathbb{R}^3$ enclosed by a surface S

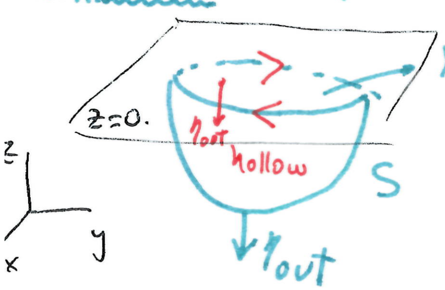
Thm: $$\text{Flux}(F,S) = \iint_S F\cdot\vec\eta_{out}\,dS = \iiint_{\mathbb{R}^3} \underbrace{\text{div}(F)}_{= f_x+g_y+h_z}\,dV$$

$F = \langle f, g, h\rangle$ & f, g, h have cont 1^{st} partials.

Extension: D enclosed in between 2 surfaces

$$\text{Flux}(F,S) = \iint_{S_1} F\cdot\vec\eta_{out}\,dS - \iint_{S_2} F\cdot\vec\eta_{in}\,dS = \iiint_D \text{div}(F)\,dV.$$

Application: Compute the flux of a surface bounded by a curve C.



$$\iint_S F\cdot\eta\,dS + \iint_{\oslash} F\cdot\eta\,dS = \iiint_V \text{div}\,F\,dV$$

Note: $$\iint_{\oslash} F\cdot\vec\eta_{out}\,dS = -\iint_{\oslash} h_{(x,y,0)}\,dA.$$

$\vec\eta_{out} = \langle 0,0,-1\rangle$

$\vec F = \langle f, g, h\rangle$.

$\oslash$ region in the xy-plane! $z=0$.