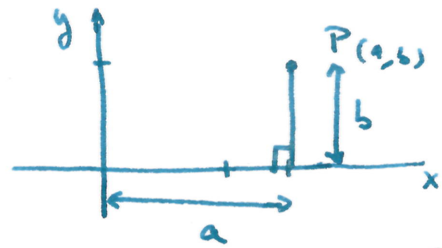
Review Midterm 1 (§ 12.1 - 13.3)

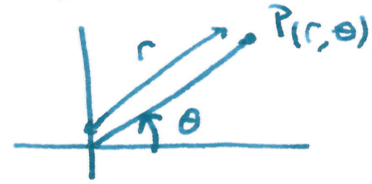
§1 Coordinate systems in $\mathbb{R}^2$ & $\mathbb{R}^3$:

• In $\mathbb{R}^2$ we have 2 coord. systems:

(1) Cartesian: $P(a,b)$ $\quad a,b$ in $\mathbb{R}$
a = x-coord
b = y-coord

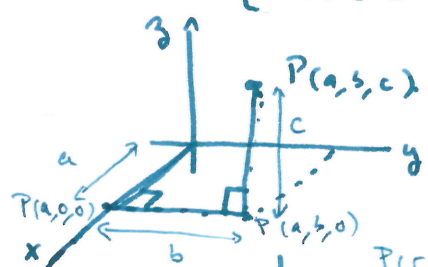


(2) Polar $P(r,\theta)$ $\quad r$ = radial coord >0
θ = angular coord
(usually $0 \leq \theta < 2\pi$)

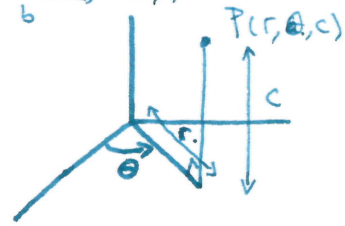


Conversion rules: $x = r\cos\theta$, $y = r\sin\theta$ & $\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \quad (x \neq 0). \end{cases}$

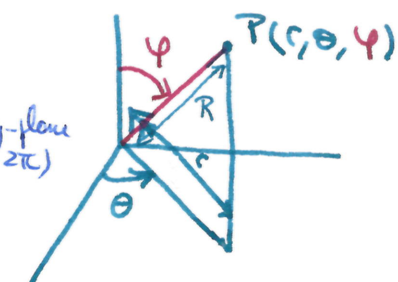
• In $\mathbb{R}^3$: (1) Cartesian $P(a,b,c)$
$\begin{cases} a = x\text{-coord} \\ b = y\text{-coord} \\ c = z\text{-coord}. \end{cases}$



(2) Cylindrical $P(r,\theta,c)$
$\begin{cases} P(r,\theta) = \text{polar coords.} \\ c = z\text{-coord.} \end{cases}$



(3) Spherical $P(R,\theta,\varphi)$
$\begin{cases} R = \text{radial coord} \\ \theta = \text{angle between } x\text{-axis (pos) \& projection to } xy\text{-plane } (0 \leq \theta < 2\pi) \\ \varphi = \text{angle w/ } z\text{-axis (positive)} \\ \qquad (0 \leq \varphi \leq \pi) \end{cases}$



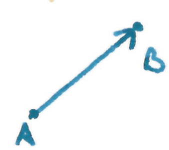
Conversion: $R = \sqrt{x^2+y^2+z^2}$
$\tan\theta = \dfrac{y}{x}$ $(x \neq 0)$, $\cos\varphi = \dfrac{z}{R}$; $\quad x = R\sin\varphi\cos\theta$, $y = R\sin\varphi\sin\varphi$, $z = R\cos\varphi$

Closed Ball w/ center $P(a,b,c)$ & radius r: $(x-a)^2+(y-b)^2+(z-c)^2 \leq r^2$, Sphere: $(x-a)^2+(y-b)^2+(z-c)^2 = r^2$.

§2 Vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$:

Vectors = data of magnitude (or length) and a direction
$\vec{v} = \langle a, b, c \rangle$ → components of $\vec{v}$
$(\langle a, b \rangle)$

Magnitude: $|\vec{v}| = \sqrt{a^2+b^2+c^2}$ $\quad (\sqrt{a^2+b^2})$



$\vec{AB} = \langle a, b \rangle$

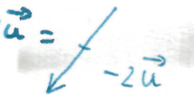
Classical representative: tail at the origin & head = components $P(a,b,c)$.

① **Addition**: Use triangle / Parallelogram Law (Eg sum of forces acting on an object)

$\vec{u} + \vec{v} =$ 

② **Scalar multiplication**: c in $\mathbb{R}$

$c\vec{u}$. direction: $\begin{cases} \text{same as } \vec{u} & \text{if } c>0 \\ \text{opposite to } \vec{u} & \text{if } c<0 \\ \text{no direction} & \text{if } c=0 \end{cases}$

Eg: $c=-2$ $\quad \vec{u}$; $-2\vec{u} =$ 

· magnitude $= |c|\,|\vec{u}|$

· subtract $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

· Satisfy 10 nice properties (Commutativity, Assoc, Distrib ...)

· Standard unit vectors $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$ $\quad \left(\begin{matrix} i = \langle 1,0\rangle \\ j = \langle 0,1\rangle \end{matrix}\right)$

· Applications = system in equilibrium $\Leftrightarrow$ sum of acting forces $=0$.

③ **Dot product** $\quad \vec{u} = \langle u_1,u_2,u_3\rangle$

$\vec{v} = \langle v_1,v_2,v_3\rangle$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$. (in $\mathbb{R}^2$: first 2 terms).

is a <u>scalar</u> in $\mathbb{R}$.

<u>Properties</u>

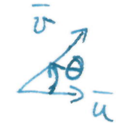
(1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

(2) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

(3) $(m\vec{u})\cdot\vec{v} = m(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(m\vec{v})$

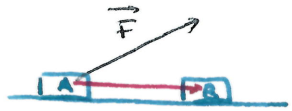
(4) $\vec{v}\cdot\vec{v} = |\vec{v}|^2$

<u>Thm</u>: $\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$ $\quad \theta =$ angle between $\vec{u}$ & $\vec{v}$. $(0 \le \theta \le \pi)$



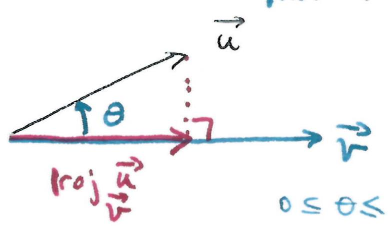
<u>Corollary</u> $\vec{u} \perp \vec{v} \iff \vec{u}\cdot\vec{v} = 0$.

<u>Applications</u>: (1)



Work of force $\vec{F}$ when moving object from A to B is $\boxed{\vec{F}\cdot\vec{AB}}$.

(2) Projections: $(\vec{v} \neq \vec{0})$



$0 \le \theta \le \pi$

$\text{proj}_{\vec{v}}\vec{u}$ = vector projection of $\vec{u}$ onto $\vec{v}$

$\text{scal}_{\vec{v}}\vec{u} = \text{comp}_{\vec{v}}\vec{u}$ = scalar comp. of $\vec{u}$ in the direction of $\vec{v}$ = <u>signed</u> magnitude of $\text{proj}_{\vec{v}}\vec{u}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}_{\text{"}\text{scal}_{\vec{v}}\vec{u}\text{"}} \underbrace{\boxed{\frac{\vec{v}}{|\vec{v}|}}}_{\text{"unit vector in the direction of } \vec{v}\text{"}}$$

Eg. $\text{scal}_{\vec{i}}\langle a,b,c\rangle = a$, $\text{scal}_{\vec{j}}\langle a,b,c\rangle = b$, $\text{scal}_{\vec{k}}\langle a,b,c\rangle = c$.

④ **Cross product** (Only in $\mathbb{R}^3$!).

$u = \langle u_1,u_2,u_3\rangle$

$v = \langle v_1,v_2,v_3\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}(u_2v_3 - u_3v_2) - \vec{j}(u_1v_3 - u_3v_1) + \vec{k}(u_1v_2 - u_2v_1).$$

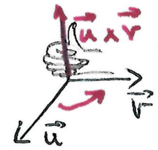
is a <u>vector</u> in $\mathbb{R}^3$.

Properties: (1) $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

(2) $(a\vec{u})\times\vec{v} = a(\vec{u}\times\vec{v}) = \vec{u}\times(a\vec{v})$.

(3) $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w}+\vec{v}\times\vec{w}$. (same for addition in 2nd term)

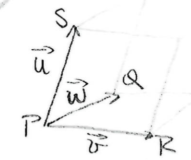
KEY prop: $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ & $\vec{v}$ & direction = right hand rule



- $|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$ θ = angle between $\vec{u}$ & $\vec{v}$ $(0\le\theta\le\pi)$
- $\vec{u}$ & $\vec{v}$ non-zero vectors are parallel $\Longleftrightarrow$ $\vec{u}\times\vec{v}=\vec{0}$.
  $(\theta = 0$ or $180°)$
- $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$
- $\vec{u}\times(\vec{v}\times\vec{w}) \neq (\vec{u}\times\vec{v})\times\vec{w}$ in general (Eg $\vec{u}=\vec{i}=\vec{v}$, $\vec{w}=\vec{j}$)

Applications: (1) Area of parallelogram w/ edges $\vec{u}$ & $\vec{v}$ $= |\vec{u}\times\vec{v}|$

(2) Volume " parallelopipe " " $\vec{u},\vec{v},\vec{w}$ $= |\vec{u}\cdot(\vec{v}\times\vec{w})|$

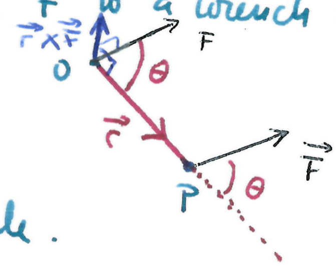


(P, Q, R, S in $\mathbb{R}^3$ are coplanar pts $\Leftrightarrow$ Vol$(\vec{PQ},\vec{PR},\vec{PS})=0$.)

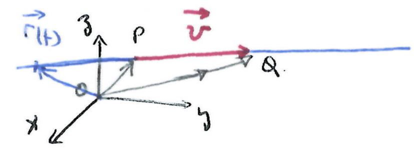
(3) TORQUE: Twist effect of applying a force $\vec{F}$ to a wrench $\vec{r}=\vec{OP}$ at the pt P is $\vec{r}\times\vec{F}$.

$\boxed{\text{Torque} = |\vec{r}\times\vec{F}| = |\vec{r}|\,|\vec{F}|\sin\theta}$

in direction = right-hand rule.



§3. Lines and planes in $\mathbb{R}^3$:

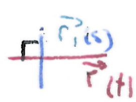


- Line through 2 distinct pts P & Q = line through pt P with direction $\vec{v}=\vec{PQ}$ $\neq\vec{0}$*

Parameterization $\vec{r}:\mathbb{R}\longrightarrow\mathbb{R}^3$ $\boxed{\vec{r}(t) = \vec{OP} + t\,\vec{v}}$ (same for line in $\mathbb{R}^2$).

Eg: $P=(1,2,3)$
$Q=(1\,1\,4)$ $\Rightarrow \vec{v}=\langle 0,-1,1\rangle$
$\vec{OP}=\langle 1,2,3\rangle$ $\vec{r}(t)=\langle t,-1+2t,1+3t\rangle$.

- Line segment between P & Q: $\boxed{0\le t\le 1}$.
- 2 lines $\vec{r}(s), \vec{r}_1(t)$ are parallel if and only if ($\Longleftrightarrow$) their directions ($\vec{v}$ & $\vec{v}_1$) are parallel
  ——— perpendicular $\Longleftrightarrow$ $\vec{v}\perp\vec{v}_1$ and $\vec{r}(s)$ & $\vec{r}_1(t)$ meet
  ——— skew $\Longleftrightarrow$ not parallel & don't meet.

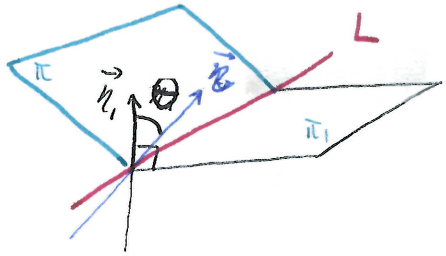


- Plane through 3 distinct non-colinear points P, R, Q = plane through pt P with 2 directions $\vec{v}=\vec{PR}$, $\vec{w}=\vec{PQ}$ $\neq\vec{0}$*

Parametric form: $\vec{r}(s,t) = \vec{OP} + s\vec{v} + t\vec{w}$

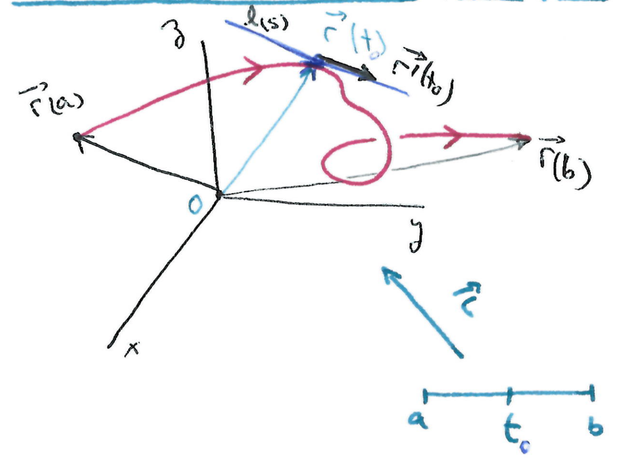
Equation: $P=(x_0,y_0,z_0)$, $\vec{n}=\vec{v}\times\vec{w}$ normal dir: $\boxed{\langle x,y,z\rangle\cdot\vec{n} = \langle x_0,y_0,z_0\rangle\cdot\vec{n}}$

- 2 planes are parallel ($\Longleftrightarrow$ their normal directions are parallel) or they meet at a line L
- angle between planes = acute angle θ between the normal directions



- The distance from a pt P to a plane π is achieved at a pt Q = meet of π with the line through P & direction $\vec{v}$ = normal direction to π

§4. Parametric curves, length.



$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle \quad (a \le t \le b)$.
vector-valued function.
(Eg: $\vec{r}(t) = \langle \cos t, \sin t\rangle \quad (0 \le t \le 2\pi)$: unit circle)

Notions: • domain = max. set where f, g & h are all defined.
- limits, continuity, derivatives, antiderivatives, integrals are done componentwise.
- tangent line at head of $\vec{r}(t_0)$
  $\ell(s) = \vec{r}(t_0) + s\,\vec{r}'(t_0)$

$\left.\frac{d\vec{r}}{dt}\right|_{t=t_0} = \vec{r}'(t_0)$ = rate of change of $\vec{r}(t)$ at time t_0. $= \langle f'(t_0), g'(t_0), h'(t_0)\rangle$.
= velocity at time t_0

- Speed $= |\vec{r}'(t)|$
- Acceleration $= \vec{r}''(t) = \frac{d}{dt}(\vec{r}')$ (iterate derivation twice)
- Derivative Rules: • constant, sum, prod rule (componentwise)
  - Chain rule $\frac{d}{dt}(\vec{u}(p(t))) = \underbrace{\vec{u}'(p(t))}_{\text{vector}} \cdot \underbrace{p'(t)}_{\text{scalar}}$
    $p: \mathbb{R} \longrightarrow [a,b]$
  - Dot / Cross Prod Rule $\frac{d}{dt}(\vec{u}(t) \underset{\times}{\cdot} \vec{v}(t)) = \vec{u}'(t) \underset{\times}{\cdot} \vec{v}(t) + \vec{u}(t) \underset{\times}{\cdot} \vec{v}'(t)$

- Prop $\frac{d}{dt}|\vec{r}(t)| = \dfrac{\vec{r}'(t)\cdot\vec{r}(t)}{|\vec{r}(t)|}$ if $\vec{r}(t) \neq \vec{0}$.

Application: If $|\vec{r}(t)|$ is constant (as a function: $[a,b] \to \mathbb{R}$) then $\vec{r}(t) \perp \vec{r}'(t)$
Eg $\vec{r}(t)$ with image in a circle / sphere.

- Antiderivative: $\vec{R}(t)$ st $R'(t) = \vec{r}(t)$ (well defined up to constant)
  Then: $\int_a^t r(p)\,dp = \vec{R}(t) - \vec{R}(a)$ $\quad R(t) = \int r(p)\,dp + \vec{C}$.
  (Fund. Thm of calculus) (compute R componentwise)
  • Application = see (*)

- Length : $L = \int_a^b |\vec{r}'(t)|\, dt$. [Application: compute $\vec{r}(t)$ from $\vec{a}(t)$ given $\vec{r}(a)$ & $\vec{r}'(a)$ (initial position & velocity)]
- Arc length (length travelled at time t) $s(t) = \int_a^t |\vec{r}'(p)|\, dp$.

  Can write $t = t(s)$ & get arc length reparameterization of $\vec{r}(t)$.

  (HARD in general!) → time as a function of length s travelled

  $$\vec{r}_1(s) = \vec{r}(t(s)) : [0,L] \longrightarrow \mathbb{R}^3 .$$

§5 Motion in the plane & in space :

① Straight line = constant velocity $<a,b,c>$ & starting at (x_0, y_0, z_0)

$$\vec{r}(t) = <x_0 + at,\ y_0 + bt,\ z_0 + ct> \quad \text{for } t \geq 0 .$$

② Circular motion : fixed center (eg (a,b)) & radius ρ in $\mathbb{R}_{>0}$ ..

$$\vec{r}(t) = <\rho\cos t + a,\ \rho\sin t + b> \qquad \text{for } 0 \leq t \leq 2\pi .$$

$$\vec{r}'(t) \perp (\vec{r}(t) - <a,b>) \quad , \quad \vec{r}''(t) = -(\vec{r}(t) - <a,b>) .$$

③ Newton's laws $\boxed{m \cdot \vec{a}(t) = \vec{F}}$

m = mass (scalar)

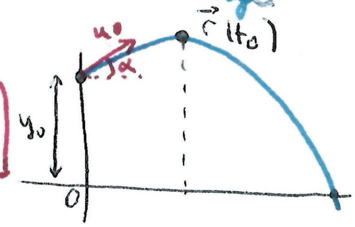
$\vec{a}(t)$ = acceleration (vector)

$\vec{F}$ = force (vector)

- Motion in a gravitational field : $g = 9.8 \frac{m}{s^2} = 32 \,ft/s^2$.

$$\begin{cases} \vec{a}(t) = <0,-g> & (\text{or } = <0,0,-g> \text{ if motion in space}) \\ \vec{v}(0) = <v_1, v_2> & (= <u_0\cos\alpha, u_0\sin\alpha> \quad u_0 = \text{initial speed}, \alpha = \text{initial angle}) \\ \vec{r}(0) = <0, y_0> \end{cases}$$

integrate twice : $\boxed{\vec{r}(t) = <v_1 t,\ -\frac{gt^2}{2} + v_2 t + y_0>}$



- Maximal height : max y-component of $\vec{r}$ $= y_0 + \frac{v_2^2}{2g}$.

  ↳ Achieved at time t_0 = $\vec{r}'(t_0)$ has y-comp $= 0$.

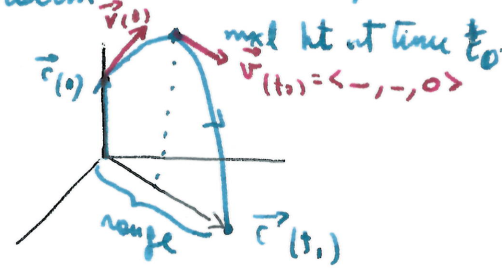
- Time of flight : time t_1 when object hits the ground $\vec{r}(t_1)$ has y-comp $= 0$

  Solve quadratic eqn! $t_1 = \dfrac{v_2 + \sqrt{v_2^2 + 2gy_0}}{g}$

- Range : horizontal distance travelled $=$ x-comp of $\vec{r}(t_1)$ $= v_1 t_1$.

- Motion with other forces (eg crosswind, spins, etc) acting:

$$\begin{cases} \cdot \text{ acceleration } = \vec{a}(t) = \langle 0, 0, -g \rangle + \text{acceleration due to other forces.} \\ \cdot \text{ initial position } \vec{r}(0) = \langle 0, 0, z_0 \rangle. \\ \cdot \text{ " velocity } \vec{v}(0) = \langle a, b, c \rangle \end{cases}$$

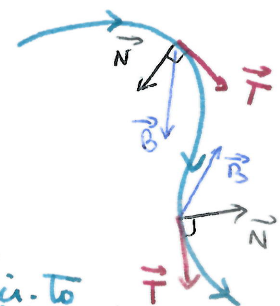


- Integrate twice to get Trajectory $\vec{r}(t)$
- Maximal height : z-comp of $\vec{v}(t_0) = 0$. $\Rightarrow$ max ht $= z$-comp of $\vec{r}(t_0)$ at time t_0.
- Time of flight : Time t_1 when object hits the ground: z-comp of $\vec{r}(t_1) = 0$
- Range $= |\vec{r}(t_1)|$

§6 Curvature, Torsion, TNB-frame:

- Unit tangent vector $\boxed{\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$ (if $\vec{r}\,'(t) \neq \vec{0}$)
- Unit normal vector : $\vec{N}(t) = \dfrac{\frac{d\vec{T}}{dt}(t)}{\left|\frac{d\vec{T}}{dt}(t)\right|} = \boxed{\dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}}$

$$\begin{cases} \cdot\ \vec{T}(t) \perp \vec{N}(t) \text{ for all } t \\ \cdot \text{ points in the direction in which the curve is turning} \end{cases}$$



- Unit binormal vector $\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$ = normal dir. to (TN) osculating plane.

$\vec{B}(t) \perp \vec{T}(t)$, $\vec{B}(t) \perp \vec{N}(t)$. $\rightsquigarrow$ they form a frame (like $\vec{i}, \vec{j}, \vec{k}$).

Two scalars:

① Curvature $\boxed{\kappa(t) = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} \quad \left(= \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \text{ for paths in } \mathbb{R}^3 \right)}$

(Eg circle of radius a : $\kappa(t) = \frac{1}{a}$ is constant, line has curvature $= 0$)

② Torsion : $\dfrac{d\vec{B}}{ds} = \dfrac{1}{|\vec{r}\,'(t)|}\dfrac{d\vec{B}}{dt} = -\tau(t)\vec{N}(t)$ where $\boxed{\tau(t) = \text{torsion} = \dfrac{-1}{|\vec{r}\,'(t)|}\dfrac{d\vec{B}}{dt}(t) \cdot \vec{N}(t)}$

Components of acceleration:

$\vec{a}(t) = \underbrace{a_T(t)}_{\text{scalar}}\vec{T}(t) + \underbrace{a_N(t)}_{\text{scalar}}\vec{N}(t)$ where $a_T(t) = \vec{a}(t) \cdot \vec{T}(t) = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}|\vec{r}\,'(t)|$

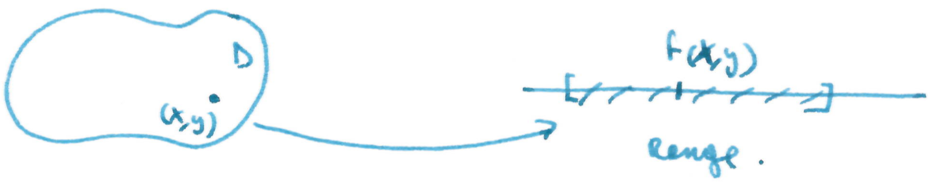
$a_N(t) = \vec{a}(t) \cdot \vec{N}(t) = \kappa(t)\,|\vec{r}\,'(t)|^2.$

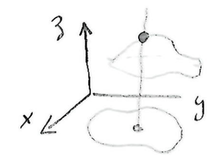
§ 7 Cylinders, Traces, quadratic surfaces.

- Draw them: using traces xy-traces :: surface intersect ($z = z_0$ plane)
- quadratic surfaces: given by a polynomial in x, y, z of degree 2
  ↳ 6 standard forms (Table 13.1).
- cylinder: move a line l along a curve C (e.g. standard cylinder)

§ 8 Functions of 2 variables:

$z = f(x, y)$ domain D, range in $\mathbb{R}$.





- Graph $= \{(x, y, f(x,y)) : (x,y) \text{ in } D\}$ surface in $\mathbb{R}^3$.
- V. Line Test: Every vertical line through D meets $z = f(x,y)$ at exact 1 pt.
- Level curves: $z_0 = f(x,y)$ for fixed z_0. (points (x,y) in D that are mapped to z_0 under f)
- Limits at P_0: For every $\varepsilon > 0$, there exists $\delta = \delta_{(\varepsilon)} > 0$ such that if P lies in D and $0 < |\overrightarrow{PP_0}| < \delta$, then $|f(P) - L| < \varepsilon$. Write $L = \lim_{P \to P_0} f(P)$.

$f(B(P_0, \delta))$ lies in the interval $(L-\varepsilon, L+\varepsilon)$

Key: Curve Test: The limit doesn't exist if we find 2 curves along which the limits are different.

If the limit L exists, then $L = \lim_{\substack{(x,y) \to P_0 \\ (x,y) \text{ in the curve } C}} f(x,y)$ for EVERY curve C passing through P_0 lying inside D.

Typical paths:
- $y = mx$ ($P_0 = (0,0)$) for m in $\mathbb{R}$. (Linear)
- $y = mx^2$ " " " " ". (Quadratic).

- Limits of sums, products, differences: same rules as in 1-var. case.

Thm: $g : \mathbb{R} \longrightarrow \mathbb{R}$ cont. at $f(a,b)$, then $g(f(x,y))$ is continuous at (a,b).
$f : D \longrightarrow \mathbb{R}$ " " (a,b)

Def: $f : D \longrightarrow \mathbb{R}$ is continuous at (a,b) provided:

(1) f defined at (a,b) (~~so~~ (a,b) in D)

(2) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists

(3) The limit equals $f(a,b)$.