Review Midterm 2 (§ 13.4 – 14.5)

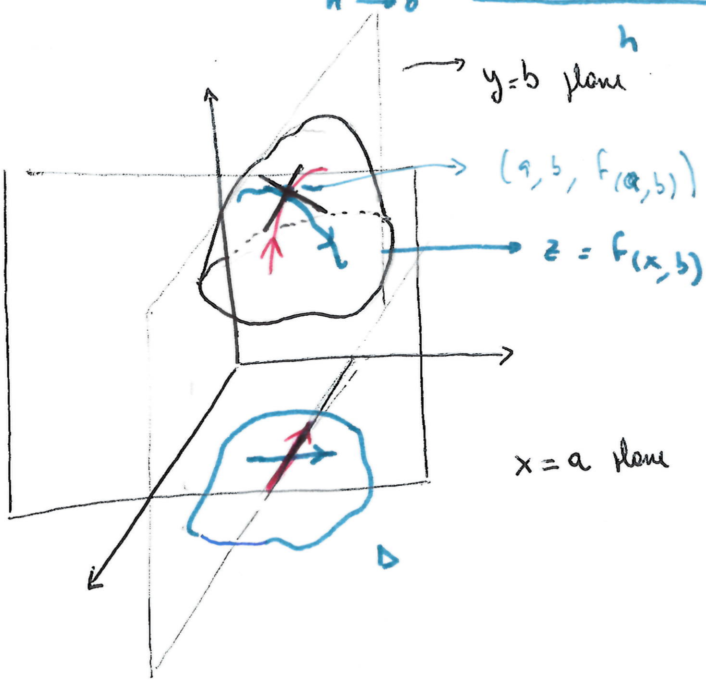
§1 Partial derivatives:

Fix $\underset{(x,y)}{f = f} : D \longrightarrow \mathbb{R}$ a function of 2 variables & (a,b) an interior pt in D

Def: $f_x(a,b) = \lim\limits_{h \to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$ = rate of change of $g(x) := f(x,b)$ at $x = a$ (g defined on $(a-\varepsilon, a+\varepsilon)$, for some $\varepsilon > 0$)

$f_y(a,b) = \lim\limits_{h \to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$ = rate of change of $q(y) := f(a,y)$ at $y = b$ (g is defined on $(b-\varepsilon, b+\varepsilon)$ for some $\varepsilon > 0$)

Idea:



- Draw the graph of $g(x)$ in the $y=b$ plane
- ———— $q(y)$ — $x = a$ plane

rates of change give the slopes of the corresponding tangent lines.

- We compute $f_x = \dfrac{\partial f}{\partial x}$ by thinking of y as a constant.

  $f_y = \dfrac{\partial f}{\partial y}$ ———— x ————.

Eg: $f(x,y) = x^2 + xy \Rightarrow \begin{cases} f_x = 2x + y. \\ f_y = x. \end{cases}$ they are again functions, defined on a subset of the domain of f.

- Higher order derivatives: "Take partials in succesion"

Eg: $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y} f_x = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$

new function

read from left to right

from right to left

Eg above: $f_{xx} = 2$, $f_{xy} = 1$, $f_{yx} = 1$, $f_{yy} = 0$.

Theorem: (Cross derivative Test) $f_{xy} = f_{yx}$ if BOTH f_{xy} & f_{yx} are continuous functions.

Application: can use this test to check continuity: if $f_{xy} \neq f_{yx}$ then one of them is not continuous (Exercises in Recitation 6 & Lecture XV)

- Typical situation: f defined as a partition function $f(x,y) = \begin{cases} \text{——} & \text{if } (x,y) \text{ in } R \\ \text{——} & \text{otherwise.} \end{cases}$

Def Differentiability at (a,b): $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$. equals.

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}_{\text{ERROR TERM.}}$$

where $\varepsilon_1, \varepsilon_2$ are functions depending ONLY on Δx & Δy (where a, b are fixed)

and $(\varepsilon_1, \varepsilon_2) \xrightarrow[(\Delta x, \Delta y) \to (0,0)]{} 0$

Theorem: If f_x & f_y are continuous at (a,b), then f is differentiable at (a,b).

- Warning: existence of f_x & f_y is not enough to get differentiability!

Theorem: f differentiable at (a,b), then f is continuous at (a,b).

§2 Linear approximation:

If f is differentiable, then if (x,y) are close to (a,b) we can approximate

$$f(x,y) \approx f(a,b) + \underbrace{f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{=: df(a,b)}.$$

(error term is small).

Note (RHS) is linear in x & y and is called the linear approximation

- For functions of 3 variables:

$$f(x,y,z) \approx f(a,b,c) + \underbrace{f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}_{=: df(a,b,c).}$$

$\begin{cases} \cdot \dfrac{dx}{x} = \dfrac{x_E - a}{x} = \text{relative error in measuring the quantity } x \text{ near } a. \\ \cdot dx := x - a = \text{absolute ——————— } x \text{ — } a. \end{cases}$ (usually given by a percentage).

Can use these to estimate effects of measurement errors.

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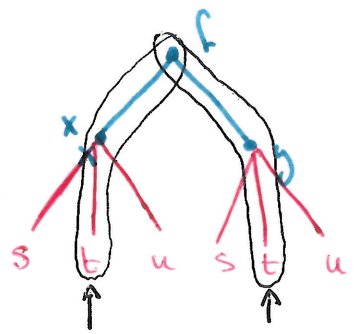
§3. Chain Rule: $f: D \longrightarrow \mathbb{R}$ $(D \text{ in } \mathbb{R}^2)$

Idea. $f = f_{(x,y)}$ and $x = x(s,t,u): \mathbb{R}^3 \longrightarrow \mathbb{R}$

$y = y(s,t,u): \mathbb{R}^3 \longrightarrow \mathbb{R}$.



Theorem: f differentiable at (a,b) & $(x,y): \mathbb{R}^3 \longrightarrow D$ differentiable at (c,d,p)

where $\begin{cases} x(c,d,p) = a \\ y(c,d,p) = b \end{cases}$, then $g_{(s,t,u)} = f(x_{(s,t,u)}, y_{(s,t,u)})$

is differentiable at (c,d,p) and.

$$\underline{\underline{\frac{\partial g}{dt}}} = \underbrace{f_x \cdot x_t}_{\text{contribution of } x \text{ branch}} + \underbrace{f_y \cdot y_t}_{\text{contrib of } y \text{ branch}} \quad \& \text{ similarly for } \frac{\partial g}{\partial s}, \frac{\partial g}{\partial u}.$$

(Pick leaf labelled by $\underline{\underline{t}}$ & draw branches from the leaves to the root of the dependence tree. Then add the contributions of each branch).

§4 Implicit differentiation:

Theorem: $F_{(x,y)} = 0$. defines a ~~surface~~ curve. & Pick (a,b) where $F(x,y)=0$.

(Eg $x^2+y^2=10$ & $(a,b) = (1,3)$)

If the curve around (a,b) is defined by $(x, y(x))$ & $y_{(x)}$ is a differentiable function of x, then provided $F_y(a,b) \neq 0$, we conclude $\frac{dy}{dx}(a,b) = \frac{-F_x(a,b)}{F_y(a,b)}$

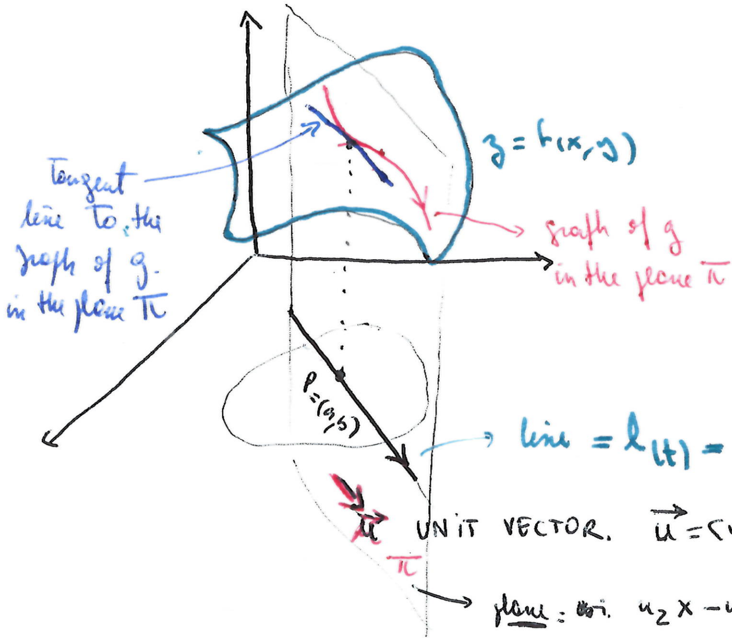
Eg: $F_{(x,y)} = x^2 + y^2 = 10$ $\quad \frac{\partial F}{\partial x} = 2x + 2y\frac{\partial y}{\partial x} = 0$ so.

$$\frac{\partial y}{\partial x} = \frac{-2x}{2y} = \frac{-F_x}{-F_y}. \quad \& \quad (2y)_{(1,3)} = 6 \neq 0 \checkmark$$

So $\frac{\partial y}{\partial x}(1,3) = \boxed{\frac{-1}{3}}$

Note: If F_x & F_y are continuous around (a,b), the condition $F_y(a,b) \neq 0$. ensures that the curve around (a,b) can be parameterized as $(x, y(x))$.

§5 Directional derivatives & the gradient: $f = f_{(x,y)} : D \longrightarrow \mathbb{R}$ (D in $\mathbb{R}^2$)



Def The <u>directional derivative</u> of f at (a,b) in direction of the <u>UNIT</u> vector $\vec{u}$ is.

$$D_{\vec{u}} f_{(a,b)} = \lim_{h \to 0} \frac{f_{(a+hu_1,\, b+hu_2)} - f_{(a,b)}}{h}$$

It's the rate of change of f when we move in the direction $\vec{u}$ (rate of $g_{(t)} = f_{(a+tu_1,\, b+tu_2)}$ at $t=0$)

line $= l_{(t)} = \langle a,b \rangle + t\vec{u} = \langle a+tu_1, b+tu_2 \rangle$

$\vec{u}$ UNIT VECTOR. $\vec{u} = \langle u_1, u_2 \rangle = \langle u_1, u_2, 0 \rangle$

plane: $u_2 x - u_1 y = u_2 a - u_1 b$.

Def: $\nabla f_{(a,b)} = \langle f_x(a,b),\ f_y(a,b) \rangle$ ($\nabla F_{(a,b,c)} = \langle f_x, f_y, f_z \rangle_{(a,b,c)}$ for functions of 3 variables)

<u>Theorem</u>: $D_{\vec{u}} f_{(a,b)} = \nabla f_{(a,b)} \cdot \vec{u}$ ($D_{\vec{u}} f_{(a,b,c)} = \nabla F_{(a,b,c)} \cdot \vec{u}$).

<u>Theorem</u>: If $\nabla f_{(a,b)} \neq \langle 0,0 \rangle$, then:

(1) f has its <u>maximum</u> rate of <u>increase</u> at (a,b) in the direction of $\nabla f_{(a,b)}$

$$\left(\vec{u} = \frac{\nabla f_{(a,b)}}{|\nabla f_{(a,b)}|} \right)$$

(2) f has its <u>maximum</u> rate of <u>decrease</u> at (a,b) in the direction of $-\nabla f_{(a,b)}$

$$\left(\vec{u} = \frac{-\nabla f_{(a,b)}}{|\nabla f_{(a,b)}|} \right)$$

(3) The directional derivative is 0 in any directional orthogonal to $\nabla f_{(a,b)}$.

[similar statement in $\mathbb{R}^3$]

<u>Applications</u> ① $\nabla f_{(a,b)}$ is orthogonal to the level curve $f_{(x,y)} = \text{constant} = f_{(a,b)}$

Tangent line to the level curve at (a,b):

$$\boxed{f_{x(a,b)}(x-a) + f_{y(a,b)}(y-b) = 0}$$

$\Rightarrow$ can predict gradients in a picture of level curves of a given function f (<u>Recitation</u>?)

② $\nabla F_{(a,b,c)}$ is orthogonal to the level surface $F_{(x,y,z)} = \text{constant} = F_{(a,b,c)}$

Tangent plane to the level surface at (a,b,c):

$$\boxed{F_{x(a,b,c)}(x-a) + F_{y(a,b,c)}(y-b) + F_{z(a,b,c)}(z-c) = 0}$$

In particular view the graph of a function $f_{(x,y)}$ as a level surface of $F_{(x,y,z)} = z - f_{(x,y)} = 0$ where $c = f_{(a,b)}$.

$\boxed{\text{Tangent plane to the graph of } f : \begin{cases} \cdot \text{ has normal } \nabla F_{(a,b,c)} = \langle F_x, F_y, F_z \rangle = \langle -f_x, -f_y, 1 \rangle \text{ at } (a,b). \\ \cdot \text{ point} = (a, b, f_{(a,b)}) \end{cases}}$

§6. Maximum / minimum problems

$f = f_{(x,y)} : D \longrightarrow \mathbb{R}$ (D in $\mathbb{R}^2$)

Analogous definitions / Thms in $\mathbb{R}^3$

Def: • (a,b) is local min of f: if the graph of f has a peak locally at $(a, b, f_{(a,b)})$

CAP =  $(a,b,f_{(a,b)})$

• (a,b) is local max of f if the graph of f has a hollow locally at $(a, b, f_{(a,b)})$

CUP =  $(a,b,f_{(a,b)})$

KEY Theorem: Assume f_x & f_y both exist at (a,b).

If f has a local max / min at (a,b), then $\nabla f_{(a,b)} = \langle 0, 0 \rangle$

Def: (a,b) is a critical point of f if either:

(1) $\nabla f_{(a,b)} = \vec{0}$

(2) At least one of $f_{x(x,y)}$, $f_{y(x,y)}$ is not defined at (a,b).

Def If (a,b) is a critical pt of f but it's NOT a local max/min, then we say (a,b) is a SADDLE POINT.

• Criterion to classify critical points: 2^{nd} Derivative Test

Thm: Assume all 2^{nd} order partials of f are continuous on a ball around (a,b) & $\nabla f_{(a,b)} = \vec{0}$. Let $D_{(x,y)} = f_{xx} f_{yy} - (f_{xy})^2$ (discriminant of f)

(1) If $D_{(a,b)} > 0$ & $f_{xx(a,b)} < 0$, then f has a local MAXIMUM at (a,b).

(2) If $D_{(a,b)} > 0$ & $f_{xx(a,b)} > 0$ ———— MINIMUM ———.

(3) If $D_{(a,b)} < 0$, then f has a SADDLE POINT at (a,b).

(4) If $D_{(a,b)} = 0$, we can't tell (anything can happen).

Thm: If f is continuous & D is closed and bounded, then f has absolute max & min values

Def • (a,b) is abs. max if $f(a,b) \geq f(x,y)$ for all (x,y) in D.

• ——— MIN ——— $\leq$ ———

• See Rec 7 for examples when D is unbounded or not closed.

Q: How to find these values?

1. Find all critical pts of f in D (candidates for local max/min).
2. Find the max / min values of f on the boundary of D.

(Ex. in Rec 7)
- → if the boundary is a union of parametric curves: eg $\triangle$ parametrize each curve & find max/min as for functions in one variable)
- → if the boundary is given as a level curve (eg $x^2+y^2=1$), use Lagrange multipliers.

3. Compare the values in 1 & 2 → the largest gives the absolute MAX value
   → " lowest ——— MIN —.

§7 Lagrange multipliers

Theorem: Fix $f(x,y): D \longrightarrow \mathbb{R}$ differentiable and D contains the curve C given by $g(x,y)=0$. Assume g is differentiable.

If (a,b) is a local max / min of f along C, then we can find λ in $\mathbb{R}$ (Lagrange multiplier)

such that:
$$\begin{cases} \nabla f(a,b) = \lambda \nabla g(a,b) & \longleftrightarrow \quad f_x(a,b) = \lambda g_x(a,b) \;\&\; f_y(a,b) = \lambda g_y(a,b) \\ g(a,b) = 0 \end{cases}$$

• If $f(x,y,z): D \longrightarrow \mathbb{R}$, then we replace C by the level surface $g(x,y,z)=0$. & the system to solve for (a,b,c,λ) is:

$$\begin{cases} \nabla f(a,b,c) = \lambda \nabla g(a,b,c) & \longleftrightarrow \quad f_x = \lambda g_x, \;\&\; f_y = \lambda g_y \;\&\; f_z = \lambda g_z \\ g(a,b,c) = 0 \end{cases}$$

• Examples in Rec 7 & Lecture XX

§8 Integrals:

- in $\mathbb{R}^2$: double integrals
- in $\mathbb{R}^3$: triple integrals

} defined using Riemann sums.

We computed them by iterations (think of each step as an integral in one-variable).

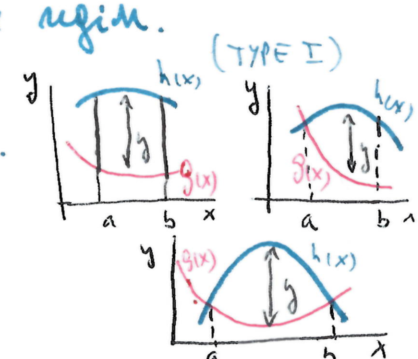
$\updownarrow$ SLICING METHOD (FUBINI's Theorem)

Thm: f continuous on a closed & bounded region, then f is integrable.

- Typical:
  - Area of a closed & bounded region R in $\mathbb{R}^2$: $\text{Area}(R) = \iint_R 1\, dA$.
  - Volume of a closed & bounded solid D in $\mathbb{R}^3$: $\text{Vol}(D) = \iiint_D 1\, dV$.
  - Mass of R $= \iint_R \rho_{(x,y)}\, dA$ ($\rho_{(x,y)}$ = density function ≥ 0)
  - Mass of D $= \iiint_D \rho_{(x,y,z)}\, dV$ ($\rho_{(x,y,z)}$ = ———— ≥ 0)

- For iterated integrals: the order of integration depends on the region.

Eg: ① $R = \{(x,y) \mid a \leq x \leq b,\ g_{(x)} \leq y \leq h_{(x)}\}$.



Then $$\iint_R f_{(x,y)}\, dA = \int_a^b \Big(\underbrace{\int_{g(x)}^{h(x)} f_{(x,y)}\, dy}_{} \Big)\, dx$$

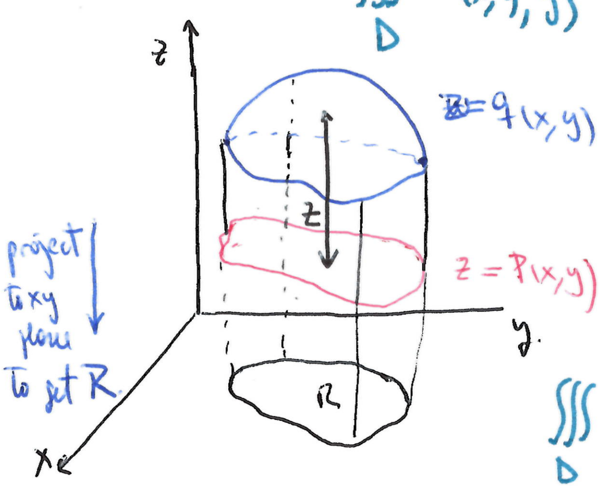
↳ compute this thinking of x as being constant.

② $D = \{(x,y,z) : a \leq x \leq b,\ g_{(x)} \leq y \leq h_{(x)},\ p_{(x,y)} \leq z \leq q_{(x,y)}\}$

Then $$\iiint_D f_{(x,y,z)}\, dV = \int_a^b \Big(\underbrace{\int_{g(x)}^{h(x)} \Big(\underbrace{\int_{p(x,y)}^{q(x,y)} f_{(x,y,z)}\, dz}_{} \Big) dy}_{} \Big)\, dz$$

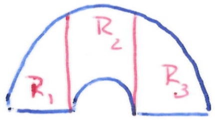
↳ compute by thinking x & y are constant

↳ compute by thinking y is constant (z has dissapeared after the 1st integral (innermost))



$$\iiint_D f_{(x,y,z)}\, dV = \iint_R \Big(\int_{p(x,y)}^{q(x,y)} f_{(x,y,z)}\, dz \Big)\, dA.$$

- If R is not a nice region to integrate, we can decompose it into nice non overlapping regions we can integrate

Eg: $R =$ 

$$\iint_R = \iint_{R_1} + \iint_{R_2} + \iint_{R_3}$$

& R_1, R_2, R_3 are type I regions

- Same idea for solids D.

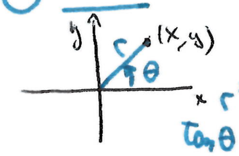
- Useful coordinate changes:

① Polar : $\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \end{cases}$

$\mathbb{R}^2_{(r\theta)} \longrightarrow \mathbb{R}^2_{(xy)}$

$S \longrightarrow R$

Eg: R = circle sector
= annulus, ...



$r^2 = x^2+y^2$

$\tan\theta = y/x$

in (r,θ)-coords in (xy)-coordinates.

$$\iint_R f_{(x,y)}\, dA_{(x,y)} = \iint_S \underbrace{f_{(r\cos\theta,\, r\operatorname{sen}\theta)}} \, r \, dA_{(r,\theta)}.$$

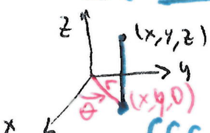
$dA_{(r,\theta)} = dr\, d\theta$ or $d\theta\, dr$.

↳ evaluate f at $x = r\cos\theta$
$y = r\operatorname{sen}\theta$.

② Cylindrical. $\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \\ z = z \end{cases}$

$\mathbb{R}^3_{(r,\theta,z)} \longrightarrow \mathbb{R}^3_{(x,y,z)}$.

$W \longrightarrow D$.



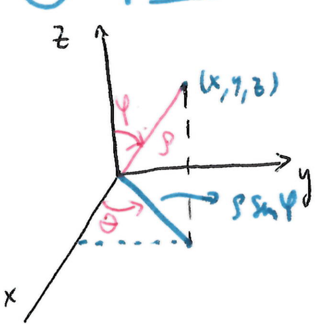
$$\iiint_D f_{(x,y,z)}\, dV = \iiint_W f_{(r\cos\theta,\, r\operatorname{sen}\theta,\, z)} \, r \, dV_{(r,\theta,z)}.$$

$dV_{(r,\theta,z)} = dr\, d\theta\, dz$ (or a reordering of these 3 terms).

③ Spherical $\begin{cases} x = (\rho \sin\varphi)\cos\theta \\ y = (\rho \operatorname{sen}\varphi)\cos\theta \\ z = \rho\cos\varphi \end{cases}$

$\mathbb{R}^3_{(\rho,\varphi,\theta)} \longrightarrow \mathbb{R}^3_{(x,y,z)}$

$W \longrightarrow D$.



$\begin{cases} \rho^2 = x^2+y^2+z^2 \\ \tan\theta = \dfrac{y}{x} \\ \cos\varphi = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$

$$\iiint_D f_{(x,y,z)}\, dV = \iiint_W f_{(\rho\sin\varphi\cos\theta,\, \rho\sin\varphi\cos\theta,\, \rho\cos\varphi)} \cdot \rho^2\sin\varphi \, dV_{\rho,\varphi,\theta}$$

$dV_{(\rho,\varphi,\theta)} = d\rho\, d\varphi\, d\theta$. (or a reordering of these 3 terms)

Hardest task: Find S & W & a nice description (Recitation 9)