SOLUTIONS

Midterm 1
Math 2153 - Calculus III (Section 10)

Prof. Cueto

Friday Feb. 19th 2016

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.

- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five minutes** to complete the exam.

- Do not forget to write your full name and Section number in the space provided below and on the bottom of the last page.

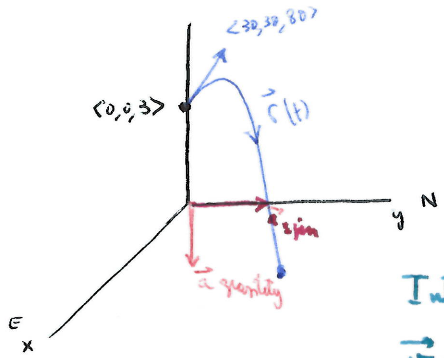
Full Name (Print): ______________________

Section number: ______________________

Enjoy the exam, and good luck!

Exercise 1. **[10 points]** A baseball is hit $3\,ft$ above home plate with an initial velocity of $\langle 30, 30, 80\rangle$ ft/s. The spin on the baseball produces a horizontal acceleration of the ball of $5\,ft/s^2$ in the northward direction.

a) Find the position vectors for all time $t \geq 0$. (Recall: $g = 32\,ft/s^2$)



$\vec{a}(t) = \vec{a}_{gravity}(t) + \vec{a}_{spin}(t)$

$= \langle 0, 0, -32\rangle + \langle 0, 5, 0\rangle = \langle 0, 5, -32\rangle$

Initial position: $\vec{r}(0) = \langle 0, 0, 3\rangle$

Initial velocity $\vec{v}(0) = \langle 30, 30, 80\rangle$

Integrate twice to get the position vector $\vec{r}(t)$.

- $\vec{v}(t) = \int \vec{a}(t)\, dt = \langle 0, 5t, -32t\rangle + \vec{C}$

  $\langle 30, 30, 80\rangle = \vec{v}(0) = \vec{0} + \vec{C}$, so $\vec{v}(t) = \langle 30, 5t + 30, 80 - 32t\rangle$

- $\vec{r}(t) = \int \vec{v}(t)\, dt = \langle 30t, \frac{5}{2}t^2 + 30t, 80t - 16t^2\rangle + \vec{C}_2$

  $\langle 0, 0, 3\rangle = \vec{r}(0) = \vec{0} + \vec{C}_2$, so

$$\boxed{\vec{r}(t) = \langle 30t, \tfrac{5}{2}t^2 + 30t, 3 + 80t - 16t^2\rangle}$$

b) What is the maximum height reached by the ball?

The max. height is reached at a time t_0 when the z-comp of $\vec{v}(t_0)$ is 0.

t_0 satisfies: $80 - 32t_0 = 0 \Rightarrow t_0 = \frac{80}{32} = \frac{5}{2}$

The height is the z-comp of $\vec{r}(t_0) = 3 + 80\frac{5}{2} - 16\left(\frac{5}{2}\right)^2$

$= 3 + 200 - 4.25 = \boxed{103\ ft}$

Exercise 2. [~~2~~30 **points**] True/False. Justify your answer with a proof if true or a counterexample if false.

[5 points]
a) For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, we have

$$|\vec{u} - 2\vec{v}| = |\vec{u}| + 4|\vec{v}|.$$

$\boxed{\text{FALSE}}$ Pick $\vec{u} = \vec{v} = \langle 1, 1 \rangle$,

then $u - 2\vec{v} = \langle -1, -1 \rangle$ $|\vec{u} - 2\vec{v}| = \sqrt{1+1} = \sqrt{2}$.

$$\left.\begin{array}{l} |\vec{u}| = \sqrt{2} \\ |\vec{v}| = \sqrt{2} \end{array}\right\} \quad |\vec{u}| + 4|\vec{v}| = 5\sqrt{2}$$

So the two expressions are different!

[5 points]
b) The planes with equations $2x - 3y + z = 9$ and $4x - 6y + 2z = 5$ are parallel.

$\boxed{\text{TRUE}}$ Two planes are parallel when their ~~equal~~ normal directions are.

$\vec{n}_1$ to plane $2x - 3y + z = 9$ is $\langle 2, -3, 1 \rangle$

$\vec{n}_2$ " " $4x - 6y + 2z = 5$ is $\langle 4, -6, 2 \rangle$

And $\vec{n}_2 = 2 \cdot \vec{n}_1$ so their are parallel.

c) [10 points]

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} = 1$$

$\boxed{\text{TRUE}}$ We define $f(x,y) = x^2+y^2$ is continuous in $\mathbb{R}^2$ because it is a polynomial.

We define $g(u) = \begin{cases} \dfrac{e^u-1}{u} & u \neq 0 \\ 1 & u = 0 \end{cases}$ is continuous in $\mathbb{R}$

(It's a rational function with non-vanishing denominator for $u \neq 0$ & for $u=0$, use L'Hôpital $\lim_{u\to 0} \frac{e^u-1}{u} \underset{0/0 \text{ indet}}{=} \lim_{u\to 0} \frac{e^u}{1} = \frac{e^0}{1} = 1$ ✓)

e^u is cont.

Our function equals $g(f(x,y))$, which is then continuous.

In particular $\lim_{(x,y)\to(0,0)} g(f(x,y)) = g(f(0,0)) = g(0) = 1$ as we wanted to show.

d) [10 points]

$$\lim_{(x,y)\to(0,0)} \frac{16\, y\, x^{3}}{x^6+y^2} = 0$$

$\boxed{\text{FALSE}}$ We find a path along which the limit is not 0.

Pick $y = x^3$ $\rightsquigarrow$ parametrization $\vec{r}(t) = \langle t, t^3 \rangle$. goes to $\langle 0,0\rangle$ when $t \to 0$.

Then $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \vec{r}(t)}} = \lim_{t\to 0} \frac{16\, t^3 t^3}{t^6+t^6} = \lim_{t\to 0} \frac{16 t^6}{2t^6} = 8 \neq 0$.

The Path Test says the limit ~~do~~ not 0 (in fact, it doesn't exist since along the x axis the limit is 0).

Exercise 3. **[18 points]** Consider the planes $\pi_1:\ 2x - y + 3z = 1$ and $\pi_2:\ x + 5y + z = 3$.

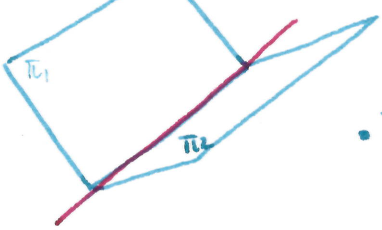
a) Show that the two planes are perpendicular to each other, i.e., their normal vectors are perpendicular.

$\vec{\eta_1}$ = normal dir to π_1 = $\langle 2, -1, 3\rangle$

$\vec{\eta_2}$ = " " " π_2 = $\langle 1, 5, 1\rangle$

$\vec{\eta_1} \cdot \vec{\eta_2} = 2 - 5 + 3 = 0$ so they are perpendicular.

b) Find the vector parameterization of the line where the two planes intersect.



To find the line, it suffices to pick a point P in both planes & find the direction $\vec{v}$ of the line.

• Pick P w/ x-coord = 0 & solve the 2 eqns:

$$\begin{cases} 0 - y + 3z = 1 & \Rightarrow y = 3z - 1 \\ 0 + 5y + z = 3 & \Rightarrow 5(3z-1) + z = 3 \end{cases}$$

$\Rightarrow \boxed{y = \frac{1}{2}}$

$16z = 8 \Rightarrow \boxed{z = \frac{1}{2}}$

so $P = (0, \frac{1}{2}, \frac{1}{2})$.

The direction $\vec{r}$ is perpendicular to $\vec{\eta_1}$ & $\vec{\eta_2}$ (it's a direction for both planes)

$$\text{so } \vec{r} = \vec{\eta_1} \times \vec{\eta_2} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = \vec{i}(-1-15) - \vec{j}(2-3) + \vec{k}(10+1) = -16\vec{i} + \vec{j} + 11\vec{k} = \langle -16, 1, 11\rangle$$

The line is given by $\boxed{\vec{r}(t) = \langle 0, \tfrac{1}{2}, \tfrac{1}{2}\rangle + t\langle -16, 1, 11\rangle \quad \text{for } t \text{ in } \mathbb{R}}$

c) Find a plane π_3 that is perpendicular to both π_1 an π_2, passing through the point $(2, 1, 1)$. (*Hint:* Use the previous item).

The plane must have a direction $\vec{\eta}$ perpendicular to both $\vec{\eta_1}$ & $\vec{\eta_2}$, so we can use $\vec{\eta} = \langle -16, 1, 11\rangle$ obtained in the previous item.

The point $(2, 1, 1)$ in π_3 is given, so the eqn is

$$\langle x, y, z\rangle \cdot \langle -16, 1, 11\rangle = \langle 2, 1, 1\rangle \cdot \langle -16, 1, 11\rangle = -32+1+11 = -20$$

$$\boxed{-16x + y + 11z = -20}$$

Exercise 4. **[10 points]** Consider the parametric curve $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle$.

a) Show that the curvature of $\mathbf{r}(t)$ is constant.

We use the formula $\kappa(t) = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$

$\vec{r}\,''(t) = \langle -\cos t, -\sin t, 0\rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \vec{i}\,(2\sin t) - \vec{j}\,(2\cos t) + \vec{k}\,(\sin^2 t + \cos^2 t)$$
$$= \langle 2\sin t, -2\cos t, 1\rangle$$

$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$.

So $\kappa(t) = \dfrac{\sqrt{5}}{(\sqrt{5})^3} = \boxed{\dfrac{1}{5}}$ is constant!

b) Show that the torsion of $\mathbf{r}(t)$ is constant.

We compute $\vec{T}(t) = \dfrac{\langle -\sin t, \cos t, 2\rangle}{\sqrt{5}}$

$\vec{T}'(t) = \dfrac{1}{\sqrt{5}}\langle -\cos t, -\sin t, 0\rangle \Rightarrow |\vec{T}'(t)| = \dfrac{1}{\sqrt{5}}$

So $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|} = \langle -\cos t, -\sin t, 0\rangle$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{5}}\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{5}}\left(\vec{i}\,(+2\sin t) - \vec{j}\,(2\cos t) + \vec{k}\,(+\sin^2 t + \cos^2 t)\right)$$
$$= \frac{1}{\sqrt{5}}\langle 2\sin t, -2\cos t, 1\rangle$$
(computed in item (a))

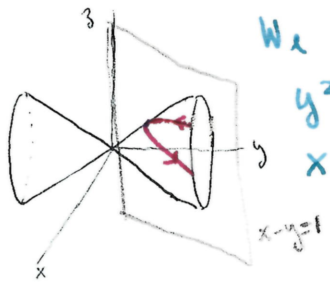
Then:

$$\frac{d\vec{B}}{ds} = \frac{1}{|\vec{r}\,'(t)|}\frac{d\vec{B}}{dt} = \frac{1}{(\sqrt{5})^2}\langle 2\cos t, +2\sin t, 0\rangle = -\tau_{(t)}\vec{N}(t) = -\tau_{(t)}\left(\langle -\cos t, -\sin t, 0\rangle\right)$$

so $\boxed{\tau_{(t)} = \dfrac{2}{5}}$ is constant!

Exercise 5. [12 points] The cone $y^2 + 3z^2 - x^2 = 0$ intersects the plane $x - y = 1$ in a parabola.

a) Give a parameterization of the parabola.



We first find an expression for each component of the points in the parabola

$y^2 + 3z^2 - x^2 = 0$ → substitute in the cone equ:

$x - y = 1 \Rightarrow \boxed{x = 1 + y}$ (*)

$0 = y^2 + 3z^2 - (1+y)^2 = y^2 + 3z^2 - 1 - y^2 - 2y$

$= 3z^2 - 1 - 2y$

so $\boxed{y = \frac{3}{2} z^2 - \frac{1}{2}}$

Replace in (*) to get x as a function of z: $x = 1 + \frac{3}{2}z^2 - \frac{1}{2} = \frac{1}{2} + \frac{3}{2}z^2$

We use z as our parameter

$$\boxed{\vec{r}(z) = \left\langle \frac{1}{2} + \frac{3}{2}z^2, \frac{3}{2}z^2 - \frac{1}{2}, z \right\rangle \text{ for } z \text{ in } \mathbb{R}}$$

is the required parametrization.

b) Find the vector parameterization of the tangent line to this parabola at the point $(2, 1, 1)$.

We check that the point $(2,1,1)$ lies in the parabola: true because it satisfies the 2 equations in the statement.

But we need to find z such that $\vec{r}(z) = (2,1,1) \Rightarrow \boxed{z = 1}$

The line $l(t)$ will be parametrized as

$$l(t) = \vec{r}(1) + t \underbrace{\vec{r}'(1)}_{\text{tangent direction to the parabola at } \vec{r}(1)}$$

$\vec{r}'(z) = \langle 3z, 3z, 1 \rangle \Rightarrow \vec{r}'(1) = \langle 3, 3, 1 \rangle$

Conclusion: $\boxed{l(t) = \langle 2, 1, 1 \rangle + t \langle 3, 3, 1 \rangle \text{ for } t \text{ in } \mathbb{R}}$

For Grader's use only:

1		2				3			4		5		TOTAL
a	b	a	b	c	d	a	b	c	a	b	a	b	