SOLUTIONS

Midterm 2
Math 2153 - Calculus III (Section 10)

Prof. Cueto

Friday April. 1st 2016

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.

- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five minutes** to complete the exam.

- Do not forget to write your full name and Section number in the space provided below and on the bottom of the last page.

Full Name (Print): ______________________

Section number: ______________________

Have fun, and good luck!

Exercise 1. [8 points]

a) [**4 points**] Find the equation of the tangent plane to the graph of the function $f(x,y) = \ln(1+xy)$ at the point $(1, 2, \ln 3)$.

• normal vector $= \langle -f_x(1,2), -f_y(1,2), 1 \rangle$

• pt $= (1, 2, f(1,2))$. Check $f(1,2) = \ln(1+2) = \ln 3$, so the point lies on the surface

$f_x(x,y) = \frac{1}{1+xy} \cdot y \rightsquigarrow f_x(1,2) = \frac{2}{3}$

$f_y(x,y) = \frac{1}{1+xy}\, x \rightsquigarrow f_y(1,2) = \frac{1}{3}$

So Tangent plane has equation:

$$-\frac{2}{3}(x-1) - \frac{1}{3}(y-2) + z - \ln 3 = 0$$

$$\boxed{z = \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

b) [**4 points**] Estimate the change in the values of the function $f(x,y) = -2y^2 + 3x^2 + xy$ when (x,y) changes from $(1,-2)$ to $(1.05, -1.9)$.

We use linear approximation:

$$\underbrace{\boxed{\Delta z = f(1.05,-1.9) - f(1,-2)}}_{\text{CHANGE}} \approx f_x(1,-2)(1.05-1) + f_y(1,-2)(-1.9-(-2))$$

$f_x = 6x + y \rightsquigarrow f_x(1,-2) = 6-2 = 4$

$f_y = -4y + x \rightsquigarrow f_y(1,-2) = 8+1 = 9$

So the change is estimated by:

$$\Delta z = 4(0.05) + 9(0.1) = 0.2 + 0.9 = \boxed{1.1}$$

Exercise 2. **[12 points]** **True/False**. Justify your answers.

a) **[4 points]** Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector. If $f(x,y)$ is differentiable at (a,b) then

$$D_{\mathbf{u}} f(a,b) = -D_{-\mathbf{u}} f(a,b).$$

TRUE

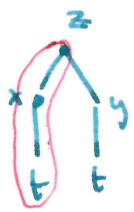
We use the formula $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

Then $D_{-\vec{u}} f(a,b) = \nabla f(a,b) \cdot (-\vec{u}) = -\nabla f(a,b) \cdot \vec{u} = -D_{\vec{u}} f(a,b)$

b) **[4 points]** If $z = (1+y)\sin(xy)$, where x and y are functions of t, then

$$\frac{\partial z}{\partial t} = (1+y)\, y \cos(xy) \frac{\partial x}{\partial t}.$$

FALSE The right hand side is only showing the contribution of the x branch of the dependence tree



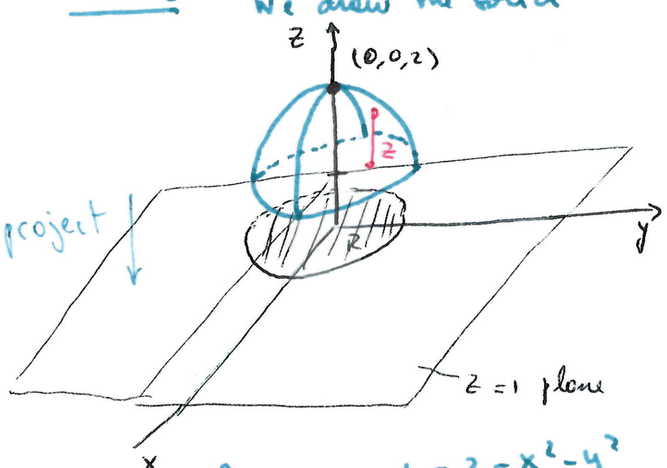
Chain rule: $\frac{\partial z}{\partial t} = z_x \frac{\partial x}{\partial t} + z_y \frac{\partial y}{\partial t}$

$$= \underbrace{(1+y)\cos(xy)\, y \frac{\partial x}{\partial t}}_{x\text{-branch contrib}} + \underbrace{\left(x(\cos xy) + \sin(xy)\right) \frac{\partial y}{\partial t}}_{y\text{-branch contrib.}}$$

c) **[4 points]** The solid bounded by the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$ has volume $\frac{\pi}{2}$.

TRUE

We draw the solid



Base: $1 = 2 - x^2 - y^2$

$x^2 + y^2 = 1$

So we get R = unit disk

In polar coordinates: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$\text{Volume} = \iint_R \left(\int_1^{2-x^2-y^2} 1 \, dz \right) dA$$

$$= \iint_R 2 - x^2 - y^2 - 1 \, dA = \iint_R 1 - x^2 - y^2 \, dA$$

polar coordinates

$$= \int_0^{2\pi} \int_0^1 (1-r^2)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left. \frac{r^2}{2} - \frac{r^4}{4} \right|_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} \, d\theta$$

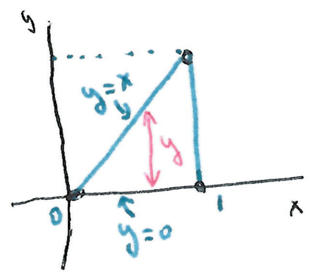
$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

So $\boxed{\text{Volume} = \frac{\pi}{2}}$

Exercise 3. [14 points]

a) [**6 points**] Compute $\displaystyle\iint_R x^2 + 3y \, dA$, where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

We draw the picture



We use Fubini & think of R as a Type I region.

$$\iint_R (x^2+3y)\, dA = \int_0^1 \left(\int_0^x (x^2+3y)\, dy \right) dx$$

$$= \int_0^1 x^2 y + \frac{3y^2}{2} \Big|_{y=0}^{y=x} dx = \int_0^1 x^3 + \frac{3}{2}x^2 \, dx$$

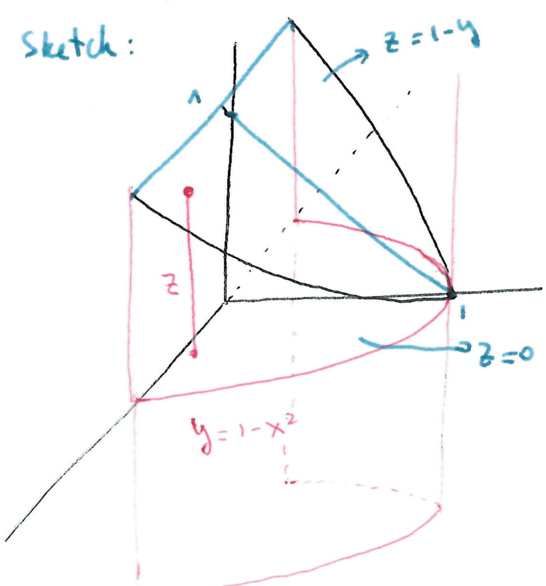
$$= \frac{x^4}{4} + \frac{x^3}{2} \Big|_{x=0}^{x=1} = \frac{1}{4} + \frac{1}{2} = \boxed{\frac{3}{4}}$$

b) [**8 points**] Consider the solid D in the first orthant bounded by the plane $z = 1 - y$ and the parabolic cylinder $y = 1 - x^2$.

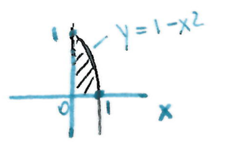
i) Sketch the solid D.

ii) Find the volume of D.

Sketch:



R = Projection =



$0 \le x \le 1$

$0 \le y \le 1 - x^2$

$$\text{Vol} = \iiint_D 1 \, dV = \iint_R \int_0^{1-y} 1 \, dz \, dA$$

$$= \int_0^1 \int_0^{1-x^2} (1-y)\, dy\, dx$$

$$= \int_0^1 y - \frac{y^2}{2} \Big|_0^{1-x^2} dx = \int_0^1 y\left(1 - \frac{y}{2}\right) \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_0^1 (1-x^2)\left(1 - \frac{(1-x^2)}{2}\right) dx = \int_0^1 (1-x^2)\left(\frac{1+x^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 1 - x^4 \, dx = \frac{1}{2}\left(x - \frac{x^5}{5}\right) \Big|_{x=0}^{x=1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{5}\right) = \frac{1}{2} \cdot \frac{4}{5} = \boxed{\frac{2}{5}}$$

So $\boxed{\text{Volume}(D) = \frac{2}{5}}$

Exercise 4. **[8 points]** Find **all** the local max/min values and saddle points of the function $f(x,y) = xye^{-x-y}$ defined on $\mathbb{R}^2$.

- The function f is differentiable up to any order, so all the critical points satisfy $\nabla f_{(x,y)} = \vec{0}$. Also, we can use the 2nd Derivative Test to determine the nature of the critical pts.
- The local max/min & saddle pts are always critical pts.

$$\nabla f_{(x,y)} = \langle f_x, f_y \rangle \underset{\substack{\uparrow \\ \text{product} \\ \text{rule}}}{=} \langle ye^{-x-y} + xye^{-x-y}(-1),\ xe^{-x-y} + xye^{-x-y}(-1) \rangle$$

$$= \underbrace{e^{-x-y}}_{\text{never } 0.} \langle y(1-x),\ x(1-y) \rangle$$

So $\nabla f_{(x,y)} = \langle 0,0 \rangle \iff y(1-x) = 0$ & $x(1-y) = 0$

$y=0$ or $x=1$ $\qquad$ $x=0$ $\quad$ $y=1$.

We have $\boxed{\text{2 critical pts } (0,0) \text{ \& } (1,1).}$

We classify them with the 2nd Derivative Test:

$$f_{xx} = y\left(e^{-x-y}(-1)(1-x) + e^{-x-y}(-1)\right) \Rightarrow f_{xx}(0,0) = 0,\ f_{xx}(1,1) = -e^{-2}$$

$$f_{yy} = x\left(e^{-x-y}(-1)(1-y) + e^{-x-y}(-1)\right) \Rightarrow f_{yy}(0,0) = 0,\ f_{yy}(1,1) = -e^{-2}$$

$$\underset{\substack{\downarrow \\ \text{they are diff so} \\ \text{continuous}}}{f_{xy} = f_{yx}} = (1-x)\left(e^{-x-y} + ye^{-x-y}(-1)\right) \Rightarrow \begin{cases} f_{xy}(0,0) = 1(1+0) = 1 \\ f_{xy}(1,1) = 0. \end{cases}$$

Write $D = f_{xx} f_{yy} - f_{xy}^2$ (discriminant of f).

Conclusion: $D_{(0,0)} = 0 \cdot 0 - 1^2 = -1 < 0 \Longrightarrow \boxed{(0,0) \text{ is a saddle pt.}}$

$$\left.\begin{array}{l} D_{(1,1)} = (-e^{-2})(-e^{-2}) - 0^2 = e^{-4} > 0 \\ f_{xx}(1,1) = -e^{-2} < 0 \end{array}\right\} \Rightarrow \boxed{(1,1) \text{ has a local MAX}}$$

Exercise 5. **[8 points]** Find all **absolute** maximum and minimum values of the function

$$f(x,y) = y^2 - 4x^2$$

on the region bounded by the ellipse $x^2 + 2y^2 = 4$.

• f is differentiable up to any order, so critical pts are those where $\nabla f_{(x,y)} = \vec{0}$

STEP 1: Look at the interior of the ellipse & search for crit. pts.

$\nabla f = \langle -8x, 2y \rangle = \vec{0} \implies \boxed{(x,y) = (0,0) \text{ only 1 crit pt}}$

STEP 2: Look at the boundary $g_{(x,y)} = x^2 + 2y^2 = 4$. ⇝ use Lagrange multipliers

$\nabla g = \langle 2x, 4y \rangle$

Find (x, y, λ) that solve $\begin{cases} -8x = \lambda 2x \iff x = 0 \text{ or } \lambda = -4 \\ 2y = \lambda 4y \iff y = 0 \text{ or } \lambda = \frac{1}{2}. \\ x^2 + 2y^2 = 4 \quad (*) \end{cases}$

• If $x = 0$ ⇝ $g(0,y) = 2y^2 = 4 \Rightarrow \boxed{y = \pm\sqrt{2}}$ & so $\lambda = \frac{1}{2}$.

• If $x \neq 0$, then $\lambda = -4$, so from the middle eqn. we get $y = 0$, &

$g(x,0) = x^2 = 4 \iff x = \pm 2$.

• 4 solutions: $(0, \sqrt{2}), (0, -\sqrt{2}), (2, 0), (-2, 0)$.

STEP 3: Compare the values from STEPS 1 & 2

$f(0, \pm\sqrt{2}) = 2 - 0 = 2$.

$f(\pm 2, 0) = -4 \cdot 4 = -16$.

$f(0,0) = 0$

$\Rightarrow$ $\boxed{\text{abs max value} = 2 \quad \text{abs min value} = -16}$

Note We know these abs. max & min values exist because f is continuous & the region is closed and bounded.

For Grader's use only:

1		2			3		4	5	TOTAL
a	b	a	b	c	a	b			