Lecture I : §1.1 Matrices & Systems of Linear Equations

§1 Introduction:

Object of study: Simplest functions of several variables (3 or more) = LINEAR
(MATH 2153 - Calc III)

Example: $X + 2Y + Z = 10$ described a plane in $\mathbb{R}^3$ with normal direction $\langle 1,2,1\rangle$, passing through $(10,0,0)$

Def: A linear equation in $\underline{n}$ unknowns is an equation that can be written as $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$ (*)

- $a_1, \ldots, a_n$ are the coefficients (fixed numbers in $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$)
- b is the constant term, or constant coefficient (also, a fixed number in $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$)
- $x_1, x_2, \ldots x_n$ are the unknowns (for $n=3$, write $x_1 = x$, $x_2 = y$, $x_3 = z$)

Why linear? Each term has degree 1 (exponent) in each unknown.

Def: A solution to (*) is a tuple of numbers $(x_1, \ldots x_n)$ satisfying (*)

Example above: $(1,5,-1)$ is a solution, $(0,0,10)$ is another sol

Non-example: $x_1 + 2\sin x_2 = 0$ ⇝ non linear in x_1, x_2 but linear in $(x_1, \sin x_2)$

First main result: Classification of the solution sets to systems of linear equations (several linear eqns stacked together).
⇝ no solution, unique solution or infinite many solutions

§2 Linear Systems

GOAL: Find simultaneous (or joint) solutions to several linear equations

We start with examples, to illustrate the general method (to be covered in Lecture 2)

<u>Examples</u>: (I) $\begin{cases} 2X + 4Y = 18 \\ X - Y = 0 \end{cases}$ (II) $\begin{cases} 2X + 4Y + Z = 18 \\ X - Y = 0 \end{cases}$

(I) 2 equations & 2 unknowns

<u>Method 1</u>: Manipulate the equations to reduce the number of terms on each one, whenever possible.

Multiply 2nd eqn by 4 & add it to 1st one:

$$\begin{cases} 2X + 4Y = 18 \\ 4X - 4Y = 0 \end{cases} \longrightarrow \begin{cases} 2X + 4Y = 18 \\ 6X = 18 \end{cases} \rightsquigarrow X = 18/6 = 3$$

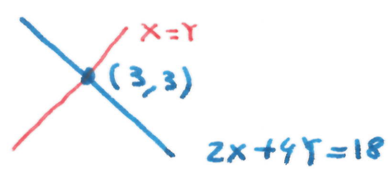
Next, substitute value for X in 1st equation & solve for Y:

$$2\cdot(3) + 4Y = 18 \rightsquigarrow 4Y = 18 - 6 = 12 \rightsquigarrow Y = 12/4 = 3$$

<u>Conclude</u>: We have a unique solution $(X,Y) = (3,3)$.

<u>Note</u>: Can always verify our answer is correct, just replace values in the system & check if the eqns are satisfied. $\begin{cases} 2\cdot 3 + 4\cdot 3 = 18 \checkmark \\ 3 - 3 = 0 \checkmark \end{cases}$

<u>Method 2</u>: Use <u>Geometry</u>!. Each equation represents a line in $\mathbb{R}^2$.
The lines are <u>not</u> parallel, so they must intersect at a unique point
This is our unique solution



(II) 2 equations, 3 unknowns.

<u>Method 1</u>: Start from 2nd eqn, replace $Y = X$ in 2nd one & solve for Z.

$$\begin{cases} 2X + 4Y + Z = 18 \\ X - Y = 0 \end{cases} \longrightarrow \begin{cases} \overbrace{2X + 4X}^{=6X} + Z = 18 \\ Y = X \end{cases} \longrightarrow \begin{cases} 6X + Z = 18 \\ Y = X \end{cases}$$

$$\begin{cases} Z = 18 - 6X \\ Y = X \end{cases}$$

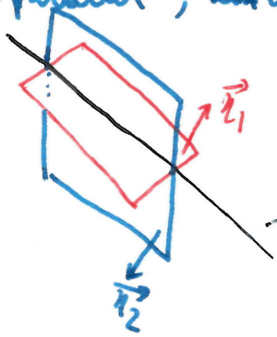
<u>Any value of X</u> will determine Y & Z, so we get <u>infinitely many solutions</u> $(X,Y,Z) = (X, X, 18-6X)$ for any $X \in \mathbb{R}$.

Can write $(x, y, z) = (0, 0, 18) + \underset{\text{parameter}}{x} \langle 1, 1, -6 \rangle$

This describes a line in $\mathbb{R}^3$ with direction $\langle 1, 1, -6 \rangle$ passing through $(0, 0, 18)$

<u>Method 2</u>: Think of equations as planes in $\mathbb{R}^3$.

2 planes with normals $\vec{n}_1 = (2, 4, 1)$, $\vec{n}_2 = (1, -1, 0)$, non parallel. So we have 2 non-parallel, distinct planes in $\mathbb{R}^3$. They must intersect along a line



→ common intersection L

. Direction of L = ?

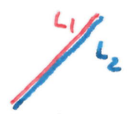
$$\Delta = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 1, 1, -6 \rangle$$

. Point in L = $(0, 0, 18)$

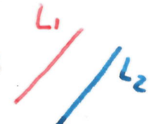
So Solution = same line we got from method 1.

<u>Summary</u> (I) 2 equations in 2 unknowns = intersection of 2 lines in $\mathbb{R}^2$

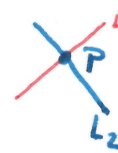
Three things can happen:

(1) 2 lines coincide 

= infinitely many solns. (1 degree of freedom)

(2) 2 lines don't coincide but are parallel : 

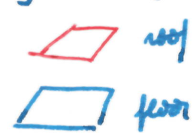
= no common solution

(3) 2 ______ & are not ______ : 

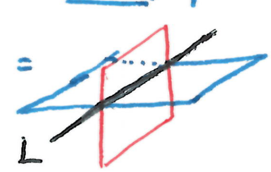
= unique Solution P.

(II) 2 equations in 3 unknowns = intersection of 2 planes in $\mathbb{R}^3$. What can happen?

(1) 2 planes coincide  = infinitely many solns (2 degrees of freedom)

(2) ______ don't coincide but are parallel = 

= <u>no</u> common solution

(3) ______ & are not ______ = 

= the common solution is the <u>line</u> L (1 degree of freedom)

<u>Note</u>: we can never have a <u>unique</u> solution if we have less equations than unknowns.