Lecture III §1.2 (cont) Echelon form & Gauss-Jordan elimination

Last time: • $m \times n$ linear systems $\longleftrightarrow$ augmented matrices $B = [A \mid \underline{b}]$ (m rows, $(n+1)$ cols)

• 3 elementary operations on equations $\longleftrightarrow$ 3 elementary operations on rows

$$\left\{\begin{array}{lll} (1) & E_p \longleftrightarrow E_q & \\ (2) & E_p \longrightarrow \alpha E_p & \text{for } \alpha \neq 0 \\ (3) & E_p \longrightarrow E_p + \lambda E_q & p \neq q \text{, any } \lambda \end{array}\right. \quad \longleftrightarrow \quad \left\{\begin{array}{lll} (1) & R_p \longleftrightarrow R_q & \\ (2) & R_p \longrightarrow \alpha R_p & \text{for } \alpha \neq 0 \\ (3) & R_p \longrightarrow R_p + \lambda R_q & p \neq q \text{, any } \lambda \end{array}\right.$$

Equivalent systems $\qquad\qquad$ Row-equivalent matrices

• Favorite shape for a matrix = echelon form & reduced echelon form

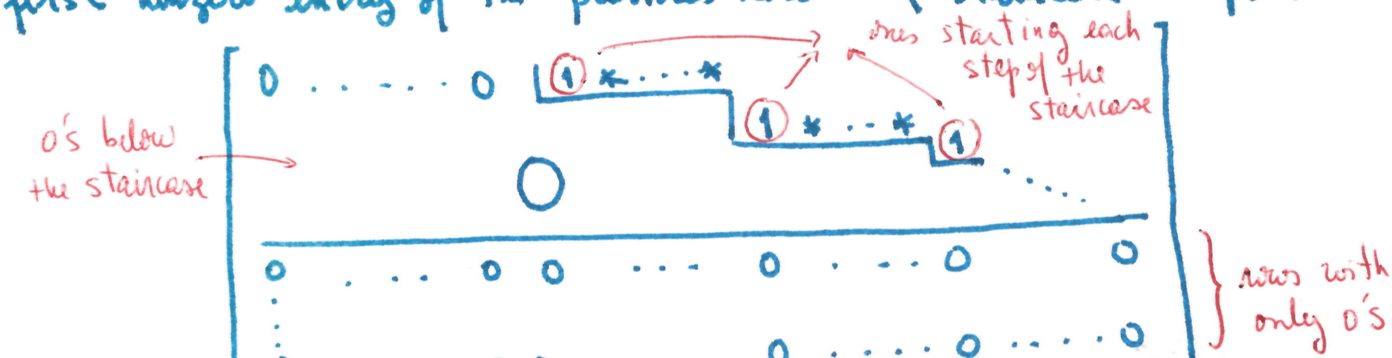
• Why? Easy to solve the associated systems!

Def: An $m \times n$ matrix A is in <u>echelon form</u> (EF) if

(1) all rows containing only 0's are grouped together at the bottom of A.

(2) in every nonzero row, the first nonzero entry (from the left) is a 1

(3) if a row is nonzero, the first nonzero entry is to the RIGHT of the first nonzero entry of the previous row ("staircase shape")



• Same for $m \times (n+1)$ matrix B.

Def: An $m \times n$ matrix A is in <u>reduced echelon form</u> (R.E.F.) if it's in E.F. & the first nonzero entry in any row is the <u>ONLY</u> nonzero entry in its column.

Example 0:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

E.F., ~~non~~ R.E.F. $\qquad\qquad$ R.E.F.

Why? REF matrices give the simplest systems to solve

Algorithm to solve systems : $B = [A\,|\,b] \xrightarrow[\text{row operations}]{} B' = [A'\,|\,b]$ with A' in R.E.F.

Example 1: $\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightsquigarrow B$

$$B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ \boxed{-1} & 1 & 7 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{} \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{array}{c} \text{FIXED!} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & \boxed{3} & 6 & 0 & 1 \end{array}\right] \end{array}$$

done with simplifying column 1, move to column 2

$$\xrightarrow[R_3 \to R_3 - 3R_2]{} \begin{array}{c} \text{FIXED} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & \boxed{3} & 3 & -8 \end{array}\right] \end{array} \xrightarrow[R_3 \to \frac{1}{3}R_3]{} \begin{array}{c} \text{FIXED!} \\ \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & \boxed{1} & -1 & 3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] \end{array}$$

ECHELON FORM
NEXT: (fix col 3, then 2)

$$\xrightarrow[R_2 \to R_2 - R_3]{} \left[\begin{array}{cccc|c} 1 & 2 & \boxed{-1} & 1 & 1 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] \xrightarrow[R_1 \to R_1 + R_3]{} \begin{array}{c} \text{FIXED!} \\ \left[\begin{array}{cccc|c} 1 & \boxed{2} & 0 & 2 & -5/3 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] \end{array}$$

done with col 3, move on to col 2

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{array}{c} \text{FIXED!} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] \end{array} = B'$$

$\rightsquigarrow$ System $\begin{cases} x_1 + 6x_4 = -13 \\ x_2 - 2x_4 = 17/3 \\ x_3 + x_4 = -8/3 \end{cases}$

dependent variables (1's in staircase)

independent variable

<u>Solutions</u>: $\begin{cases} x_1 = -13 - 6x_4 \\ x_2 = 17/3 + 2x_4 \\ x_3 = -8/3 - x_4 \\ x_4 \text{ ANY} \end{cases}$

- <u>Conclusion</u>: We have infinitely many solutions (x_4 can take <u>any</u> value in $\mathbb{R}$)
- Write $(x_1, x_2, x_3, x_4) = (-13 - 6x_4, \frac{17}{3} + 2x_4, -\frac{8}{4} - x_4, x_4)$
$$= (-13, \tfrac{17}{3}, -\tfrac{8}{3}, 0) + x_4(-6, 2, -1, 1)$$
free parameter
- Can check these are solutions of the original system:

(Eq 1) $x_2 + x_3 - x_4 = \frac{17}{3} + 2x_4 + (-\frac{8}{3} - x_4) - x_4 = \frac{9}{3} = 3$ ✓

(Eq 2) $x_1 + 2x_2 - x_3 + x_4 = (-13 - 6x_4) + 2(\frac{17}{3} + 2x_4) - (-\frac{8}{3} - x_4) + x_4$
$$= 1 + \underbrace{(-6+4+1+1)}_{=0} x_4 = 1 \checkmark$$

$$(E_q 3)\quad -x_1 + x_2 + 7x_3 - x_4 = (13+6x_4) + \left(\tfrac{17}{3}+2x_4\right) + 7\left(-\tfrac{8}{3}-x_4\right) - x_4$$
$$= \underbrace{\left(13+\tfrac{17}{3}-\tfrac{56}{3}\right)}_{=0} + \underbrace{(6+2-7-1)}_{=0}x_4 = 0\ ✓$$

Example 2

$$B = \left[\begin{array}{ccc|c} 1 & 0 & 8 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] = B' \rightsquigarrow \begin{cases} x_1 + 8x_4 = 5 \\ x_2 + 4x_4 = 7 \\ \boxed{0 = 1} \end{cases} \rightsquigarrow \text{no solution!}$$

Example 3:

$$\begin{array}{c} \\ B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \quad\quad\quad\quad\quad \uparrow \\ \quad\quad\quad\quad\quad \text{indep} \end{array} \rightsquigarrow \begin{cases} x_1 = 5 \\ x_2 = 7 - 4x_3 \\ 0 = 0 \rightsquigarrow \text{ignore it.} \end{cases}$$

Soln: $(5, 7-4x_3, x_3)$ for any x_3 (Line in $\mathbb{R}^3$)

Theorem 1: Given an $m\times(n+1)$ matrix $B = [A|b]$, there is a unique $m\times(n+1)$ matrix B' where $B' = [A'|b']$ & B' is in R.E.F, that is row equivalent to B.

We find B' using Gauss-Jordan elimination

Step 0. [Test=0 Step] If B has all zero entries, done ✓ Otherwise, B has some nonzero entry

Step 1 [find column] Pick the first (left-most) column with a nonzero entry, say col $= j$

Step 2 [swap step] Exchange rows so that j^{th} col has a nonzero entry in row one $(a_{1j} \neq 0)$

Step 3: [rescale step] If $\alpha = a_{1j}$ is the first non-zero entry in row 1, use $R_1 \to \frac{1}{\alpha} R_1$ (After this, this entry becomes 1)

Step 4: [replacement step] Replace each row R_i $(i \neq 1)$ with $R_i - a_{ij} R_1$

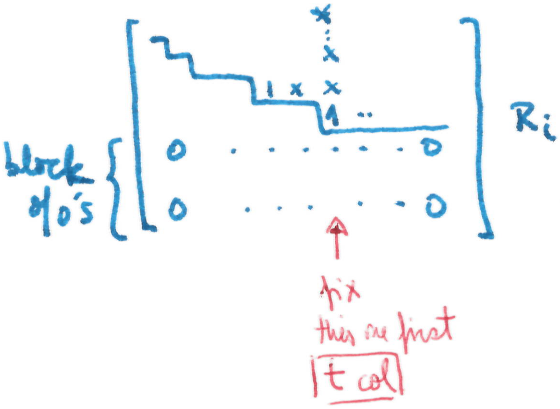
$$\rightsquigarrow \text{get}\quad \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & & 0 & 0 & * & \cdots & * \\ & & & \vdots & & & \\ 0 & \cdots & 0 & 0 & * & \cdots & * \end{array}\right] \end{array}$$

$= B_1$ is $(m-1)\times n$ matrix

$a_{ij} \longmapsto a_{ij} - a_{ij}\cdot 1 = 0$ for $i > 1$

Step 5: Repeat steps 0 through 4 for $B_1 \rightsquigarrow B_2 \rightsquigarrow B_3 \cdots$ Until we get an E.F. matrix

Step 6 From EF to REF: Work our way back to put 0's on top of each 1 starting each row. (fixing each column & moving to the left)



- Start from last nonzero row (say R_i) & for each $s<i$ do:
$$R_s \longrightarrow R_s - a_{st} R_i$$
Effect: Entries above row i on col t become 0.

- Move one row up & repeat to fix the next column to the left in the staircase.

Example 4: Solve
$$\begin{cases} x_2 - x_3 + x_4 - x_5 = 1 \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = 3 \\ -2x_2 + 2x_3 + x_4 - x_5 = 2 \\ x_2 - x_3 + 7x_4 - 7x_5 = 9 \end{cases}$$

$$B = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{\text{STEP 2}} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & \boxed{-2} & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right] = B_1$$

$$\xrightarrow[R_3 \to R_3 - (-2)R_1]{\text{STEP 4 for } B_1} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & \boxed{1} & -1 & 7 & -7 & 9 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{\text{STEP 4 for } B_1} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 6 & -6 & 8 \end{array}\right] B_2 \xrightarrow{\text{STEP 3 for } B_2} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & \boxed{6} & -6 & 8 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 6R_1]{\text{STEP 4 for } B_2} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon form EF!

Need to fix 2 columns: 4, then 2
(R3) (R2)

$$\xrightarrow[\substack{R_2 \to R_2 - R_3 \\ R_1 \to R_1 - (-1)R_3}]{\text{STEP 6 for } R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & 0 & 0 & 13/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - (-3)R_2]{\text{STEP 6 for } R_2} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 10/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

FIXED

REF!

DEPENDENT VARIABLES: x_1, x_2, x_4

Solutions to
$$\begin{cases} x_1 - 2x_3 = 10/3 \\ x_2 - x_3 = -1/3 \\ x_4 - x_5 = 4/3 \\ 0 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = 10/3 + 2x_3 \\ x_2 = -\frac{1}{3} + x_3 \\ x_4 = 4/3 + x_5 \\ x_3, x_5 \text{ ANY} \end{cases}$$

$\rightsquigarrow$ INDEP. VARS: x_3, x_5

General solution $\left(\frac{10}{3}, \frac{-1}{3}, 0, \frac{4}{3}, 0\right) + x_3(2, 1, 1, 0, 0) + x_5(0, 0, 0, 1, 1)$

2 free parameters.

Particular solutions: plug in any values for x_3 & x_5

Example: $x_3 = x_5 = 0$ gives $\left(\frac{10}{3}, \frac{-1}{3}, 0, \frac{4}{3}, 0\right)$

$x_3 = 1,\ x_5 = -1$ " $\left(\frac{16}{3}, \frac{-4}{3}, 1, \frac{1}{3}, -1\right)$

Example 5: Solve
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 + \frac{1}{2} \end{cases}$$

$$B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right] \xrightarrow[R_1 \to \frac{R_1}{2}]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & -7/2 & 0 & -7/2 \\ 0 & 35/2 & 0 & \underset{=18}{36/2} \end{array}\right]$$

$$\xrightarrow[R_2 \to \frac{-2}{7} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & +1 & 0 & +1 \\ 0 & 35/2 & 0 & 18 \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{35}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & +1/2 \end{array}\right] \xrightarrow[R_3 \to 2R_3]{} \underset{\text{E.F.}}{\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]}$$

System has no solution because 3^{rd} eqn becomes $0 = 1$.

Observation: Can solve many systems with the same coefficient matrix at once! use $B = \left[\, A \;\middle|\; \begin{array}{c} b \text{ for} \\ \text{system } 1 \end{array} \;\middle|\; \begin{array}{c} b \text{ for} \\ \text{system } 2 \end{array} \;\middle|\; \cdots \,\right]$