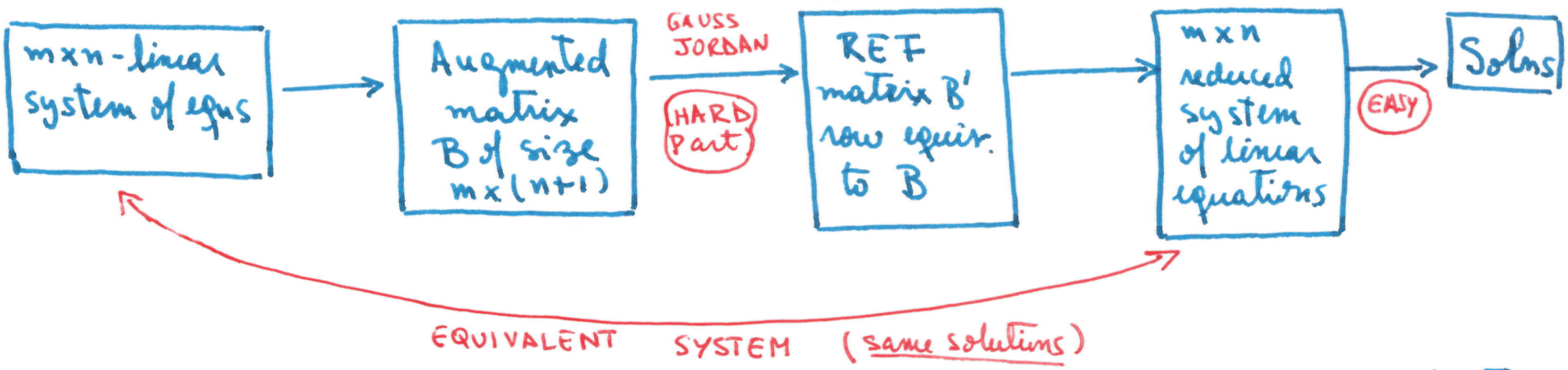
Lecture IV: §1.3 Consistent systems of linear equations

Last time: discussed Gauss-Jordan elimination algorithm: $\underset{m\times(n+1)}{B} \rightsquigarrow \underset{m\times(n+1)}{B'}$

Here, B' is the UNIQUE matrix in Reduced Echelon Form (REF) that is row-equivalent to B'.

Why? It helps solve the associated system by finding an equivalent simpler one.

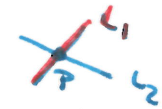


Def: A system is consistent when it admits a solution (either one or infinitely many)
If no solutions exist, we call it inconsistent, or incompatible

Example: 2×2 system $=$ 2 lines L_1 & L_2 in $\mathbb{R}^2$

① If the lines are parallel and distinct, the system is inconsistent

② In all other cases, the system is consistent: $\begin{cases} \cdot \text{ unique solution } P: \\ \cdot \text{ infinitely many}: L_1 = L_2 \end{cases}$



Solutions: $L_1 = L_2$

Extra example: page 3

§1. Solutions for consistent systems

Q1: How many solutions? Do we have any?

Q2: How do we write them down?

Write $B = [A|b] \underset{\text{row equiv}}{\sim} B' = [A'|b']$ in REF.

A1: $\boxed{\text{Consistent system} = \text{no row in } B' \text{ has the form } [0 \ldots 0 | 1] \text{ (so no equation } 0 = 1)}$

Example: $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right] \underset{R_3 \to R_3 - R_2}{\sim} B' = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \to$ inconsistent!

Consistent format: $B' =$

} rows with only 0's.

Once we know the system is consistent: first 1 in each nonzero row is a <u>dependent variable</u> (as many as nonzero rows in B'). Rest = independent variables
"<u>leading-one variables</u>" (free parameters)

<u>Example</u>: $B' = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 10 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$

dependent

consistent? YES
dependent: x_1, x_2, x_4
independent: x_3

$\begin{cases} x_1 = 10 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_4 = 4 \end{cases}$
x_3 <u>ANY</u>

$$\text{solutions} = (x_1, x_2, x_3, x_4) = (10 - 3x_3, 1 - 2x_3, x_3, 4)$$
$$= \underbrace{(10, -1, 0, 4)}_{\text{constant terms}} + x_3(-3, -2, 1, 0)$$

indep vars are parameters

Special solutions = pick values for x_3.

<u>Example</u>: $x_3 = 0$ gives $(10, -1, 0, 4)$
$x_3 = 1$ " $(7, -3, 1, 4)$
$x_3 = -1$ " $(13, 1, -1, 4)$

<u>Prop</u>: Number of independent parameters $= n - r$, where r = # nonzero rows of B'
= # dependent vars.
= "leading-one" vars.

- How many soln? (1) If <u>no indep. param</u> & system is consistent, then the solution is <u>unique</u>.
  (2) If we have indep param & the system is consistent, then we have infinitely many solutions.

<u>Def</u>: $\boxed{\text{rank}(B') = \#\text{ non-zero rows of } B'}$ ($= r$ = # dependent param for compatible systems)

⚠ Only defined like this for REF. Later on, we will see another definition
($\text{rank}(B) = \text{rank}(B')$ where $B \sim B'$ & B' in REF.)

<u>Obs</u>: $\text{rank}(B') \leq n+1$ & $\text{rank}(B') \leq n$ if the system for B' is consistent.
Also: $\text{rank}(B') \leq m$

<u>Why?</u> $\left[\begin{array}{ccc|c} 1 & & & \\ & 1 & \cdots\cdots & \\ \hline 0 & \cdots & & 0 \\ \vdots & & & \\ 0 & & \cdot & 0 \end{array}\right]$

- Cannot have more nonzero rows than:
  - the # columns of B'
  - total number of rows.
- If it is consistent, we lose one case ($[0 \cdots 0 | 1]$ is not a valid row)

<u>Thm 1</u>: If B' (in REF) is the augmented matrix of a <u>consistent $m \times n$ system</u>, then $\text{rank}(B') \leq n$ & there are $n - \text{rank}(B')$ independent variables that can be assigned arbitrary values. Moreover: $\text{rank}(B') < n$ <u>if and only if</u> there are infinitely many solns.

Extra example 2×3 system $=$ 2 planes in $\mathbb{R}^3$

$$\begin{cases} \underline{a_{11}}\, x + \underline{a_{12}}\, y + \underline{a_{13}}\, z = \underline{b_1} \\ \underline{a_{21}}\, x + \underline{a_{22}}\, y + \underline{a_{23}}\, z = \underline{b_2} \end{cases}$$

plane π_1 in $\mathbb{R}^3$.
— π_2 —.

fixed numbers

① If planes π_1 & π_2 are parallel & $\neq$ (say floor & ceiling in the classroom)
then we have no solutions = inconsistent system

② If planes are not parallel: $\pi_1 = \pi_2$ or $\pi_1 \neq \pi_2$. In both cases, we have a consistent system. In both cases, infinitely many solutions (either a plane or a line)