Lecture VI: §1.5 Matrix operations & §1.6 Algebraic properties

§1. Matrix operations:

$\boxed{\text{Scalars} = \text{real (or complex) numbers}}$

Def <u>Identity of matrices</u>: same size & same entries

<u>In symbols</u>: $A = [A_{ij}]_{i,j}$ $m\times n$ matrix & $B = [B_{ij}]_{i,j}$ $r\times s$ matrix

Then we say $\boxed{A = B}$ if
- $m = r$, $n = s$
- $A_{ij} = B_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$

<u>Examples</u>: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & \boxed{0} \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & \boxed{1} \\ 0 & 1 \end{bmatrix}$

- <u>Matrix addition & scalar multiplication</u>:

Fix A, B two $m \times n$ matrices & r in $\mathbb{R}$ (scalar)

Def: $A + B$ is an $m\times n$ matrix & $(A+B)_{ij} = A_{ij} + B_{ij}$
(sum = done entrywise)

<u>Ex</u>:
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+1 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not defined (different sizes!)

Def: $r \cdot A$ is an $m \times n$ matrix & $(rA)_{ij} = r(A_{ij})$.

Ex: $2\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 0 \\ 2\cdot 2 & 2\cdot(-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$.

§2. Vectors in $\mathbb{R}^n$ & general solutions to linear systems in vector form:

- Points in $\mathbb{R}^n$ = ordered tuple with n entries $\underline{x} = (x_1, \ldots, x_n)$
- Column Vectors of dimension n = $n \times 1$ matrices $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

<u>Ex</u>: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector of dimension 3.

- Write $\mathbb{R}^n$ for the set of all n-dimensional column vectors

Call it <u>Euclidean space</u>. It has 2 operations:
- addition
- scalar multiplication

- Write solutions to $m \times n$ linear systems in <u>vector form</u> = = add column vectors with scalar multiplication.

scalars = independent parameters

<u>Example</u>: 4x5 system

$$B' = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1+x_2-2x_3 \\ x_2 \\ x_3 \\ 2+x_5 \\ x_5 \\ 4 \end{bmatrix}$$

dependent variables

So
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 4 \end{bmatrix}}_{\text{constant vector}} + \underbrace{x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{solutions to the associated homog system}}$$

(i.e. $[A'|0]$ if $B' = [A'|b']$)

<u>Note</u>: (1) For consistent inhomogeneous systems, we will ALWAYS have a non-zero constant vector.

(2) For homogeneous ~~sys~~ the constant vector is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, so we ignore it.

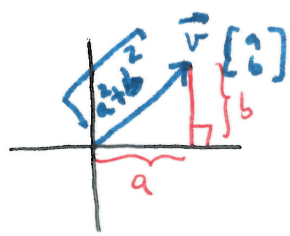
<u>New operation on $\mathbb{R}^n$</u> = scalar or dot product

<u>Def</u>: For $\underline{u}, \underline{v}$ in $\mathbb{R}^n$, we define their <u>dot product</u> as the number $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$.

<u>Example</u>: $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \rightsquigarrow \underline{u} \cdot \underline{v} = 1(-1) + 2 \cdot 4 + 3 \cdot 0 = -1 + 8 + 0 = \boxed{7}$

<u>Vector norms</u>: $\|v\| = \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2}$ (Euclidean length or norm in $\mathbb{R}^n$)

In $\mathbb{R}^2$: $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$



$\|v\| = \sqrt{a^2+b^2}$

§ 3 Matrix multiplication:

CASE 1: Matrix A of size $m \times n$ & $\underline{x}$ in $\mathbb{R}^n$

- $A\underline{x}$ in $\mathbb{R}^m$ is defined by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_n \, col_n(A).$$

Symbols: $(A \cdot x)_{i1} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{j=1}^{n} a_{ij}x_j$

Application: Write a system $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$ as a matrix product: $A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $A =$ coefficient matrix.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\rightsquigarrow$ $A\underline{y} = \begin{bmatrix} 1\cdot 1 + 2(-1) + 3\cdot 0 \\ 0\cdot 1 + 1\cdot(-1) + (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

So $\underline{y}$ is a solution to $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Example. Solve $\underbrace{a\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}}$ $= \begin{bmatrix} 1 \\ -2/3 \\ -1 \end{bmatrix}$ for a, b in $\mathbb{R}$ (Checking lin. independence / How to write linear combination)

$$\begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ -1 \end{bmatrix}.$$

Check: There is a unique solution
$a = \frac{-1}{3}$
$b = 1/3$

CASE 2: Two matrices $A, B \rightsquigarrow A \cdot B = ?$

Only defined if # cols A = # rows B.

A $m \times n$, B $n \times s$ $\rightsquigarrow$ AB is an $m \times s$ matrix

& j^{th} column of AB = $A \cdot (j^{th}$ column of $B)$.

$$\boxed{(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad \text{for } 1 \le i \le m, \ 1 \le j \le s}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \boxed{a_{i1} \ \cdots \ a_{in}} & & \\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & & b_{nj} & & b_{ns}\end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & \boxed{c_{ij}} & \\ c_{m1} & \cdots & c_{ms}\end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$ (2×3) $\quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 10 \\ 0 & 5 & 1 & 0 \end{bmatrix}$ (3×4)

$\rightsquigarrow$ $A\cdot B$ is 2×4 matrix

$$A\cdot B = \begin{bmatrix} -1 & 2\cdot 4+3\cdot 5 & 1+2+3 & 20 \\ -1 & 1\cdot 4+(-1)5 & 1-1 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 23 & 6 & 20 \\ -1 & -1 & 0 & 10 \end{bmatrix}$$

- BA is not defined $(\#\text{col}(B) = 4 \neq 2 = \#\text{rows}(A))$

Why this definition? Allows for fast substitutions (compositions of linear maps)

Ex: Combine $\begin{cases} x_1 = 3y_1 - y_2 + y_3 \\ x_2 = -3y_1 + 5y_2 \end{cases}$ & $\begin{cases} y_1 = -4z_1 + z_3 \\ y_2 = z_2 - z_3 \\ y_3 = 0 \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \& \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

So
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} -12 & -1 & 4 \\ 12 & 5 & -8 \end{bmatrix}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 = -12z_1 - z_2 + 4z_3 \\ x_2 = 12z_1 + 5z_2 - 8z_3 \end{cases}$$