Lecture VII : §1.6 Algebraic properties of matrix operations
§1.9 Matrix inverses

Recall: Last time we defined:

$\boxed{\text{size} = \#\text{ rows } \& \ \#\text{ columns}}$

- addition, scalar multiplication of matrices (operate entry by entry)
- dot product on n-dimensional column vectors ($\mathbb{R}^n = n\times 1$ matrices) & norms
- matrix multiplication : A of size $m\times \underline{\underline{n}}$ & B of size $\underline{\underline{n}}\times s$

$$AB = \begin{array}{c} \\ \left[\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \boxed{a_{i1} \ \cdots \ a_{in}} \\ a_{m1} & & a_{mn}\end{array}\right] \begin{array}{l} \\ i^{th}\text{ row} \\ \\ \end{array} \end{array} \begin{array}{c} \left[\begin{array}{ccc} b_{11} \cdots & \boxed{\begin{array}{c} b_{1j} \\ \vdots \\ b_{nj}\end{array}} & \cdots b_{1s} \\ \vdots & & \vdots \\ b_{n1} & & b_{ns}\end{array}\right] \\ j^{th}\text{ col} \end{array} = \begin{bmatrix} c_{11} & \cdots & c_{1s} \\ & \boxed{c_{ij}} & \\ c_{m1} & \cdots & c_{ms}\end{bmatrix} \quad \text{size} = m\times s$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad \text{for } \begin{array}{l} 1\le i\le m \\ 1\le j\le s\end{array}$$

$$(\underset{(\text{in } \mathbb{R}^m)}{j^{th}\text{ col } AB} = \underset{m\times n}{A} \cdot \underset{\text{in } \mathbb{R}^n}{j^{th}\text{ col } B})$$

Example:
$$\underset{2\times 2}{\begin{bmatrix} 3 & -1 \\ -3 & 5\end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix} -4 \\ 7\end{bmatrix}} = \begin{bmatrix} 3(-4)+(-1)\cdot 7 \\ (-3)(-4)+5\cdot 7\end{bmatrix} = \begin{bmatrix} -19 \\ 47\end{bmatrix} \quad 2\times 1 \quad ; \quad \underset{2\times 1}{\begin{bmatrix} -4 \\ 7\end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 3 & -1 \\ -3 & 5\end{bmatrix}} \text{ not defined}$$

Why this definition? Rules of substitution.

Ex: $\begin{cases} x_1 = 3y_1 - y_2 \\ x_2 = -3y_1 + 5y_2\end{cases}$ $\begin{cases} y_1 = -4z \\ y_2 = 7z\end{cases}$ $\rightsquigarrow$ $\begin{array}{l} x_1 = (3(-4) - 1\cdot 7)z = -19z \\ x_2 = (-3(-4) + 5\cdot 7)z = 47z\end{array}$

§1. Algebraic Properties:

Theorem 1: A, B, C $m\times n$ matrices. Then,

(1) $A + B = B + A$ [Commutative]

(2) $(A+B)+C = A+(B+C)$ [Associative]

(3) [Neutral Element] The zero matrix $\mathcal{O}$ of size $m\times n$ $(\mathcal{O}_{ij} = 0$ for all $i,j)$ satisfies $A + \mathcal{O} = \mathcal{O} + A = A$ for all $m\times n$ matrices A.

(4) [Inverses] Given A, the matrix $P = (-A_{ij})_{i,j}$ satisfies $A+P = P+A = \mathcal{O}$

Why? Addition is done on each entry separately & these properties are true in $\mathbb{R}$.

Def: The identity matrix of size $n\times n$, written I_n, has 1's in the diagonal, and 0's elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1\end{bmatrix} \text{ — diagonal}$$

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$, ...

Theorem 2: A of size $m\times n$, B of size $n\times s$, C of size $s\times q$.

(1) $\underbrace{(AB)}_{m\times s}\underbrace{C}_{s\times q} = \underbrace{A}_{m\times n}\underbrace{(BC)}_{n\times q}$ [Associative] $m\times q$ matrix

(2) α, β scalars in $\mathbb{R}$: $\alpha(\beta A) = \alpha\beta A$.

(3) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

(4) [Neutral elements]: $A = \underbrace{I_m}_{m\times m}\underbrace{A}_{m\times n} = \underbrace{A}_{m\times n}\underbrace{I_n}_{n\times n}$

Proof: Compute expressions on each side & check we have matrices of the same size with the same entries.

(1) $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$

$((AB)C)_{i\ell} = \sum_{j=1}^{s}(AB)_{ij}C_{j\ell} = \sum_{j=1}^{s}\left(\sum_{k=1}^{n} A_{ik}B_{kj}\right)C_{j\ell}$

$\underset{\text{distribute}}{=} \sum_{j=1}^{s}\sum_{k=1}^{n} A_{ik}B_{kj}C_{j\ell} \underset{\text{exchange sums}}{=} \sum_{k=1}^{n}\sum_{j=1}^{s} A_{ik}B_{kj}C_{j\ell} = \sum_{k=1}^{n} A_{ik}\underbrace{\sum_{j=1}^{s}B_{kj}C_{j\ell}}_{(BC)_{k\ell}} = (A(BC))_{i\ell}$

(2) $(\alpha(\beta A))_{ij} \underset{\text{defin. scalar prod.}}{=} \alpha(\beta A)_{ij} = \alpha\beta A_{ij} = ((\alpha\beta)A)_{ij}$.

(3) $(\alpha(AB))_{ij} = \alpha(AB)_{ij} = \alpha\sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}(\underbrace{\alpha B_{kj}}_{=(\alpha B)_{kj}}) = (A\cdot(\alpha B))_{ij}$

& also $= \sum_{k=1}^{n}\underbrace{(\alpha A_{ik})}_{(\alpha A)_{ik}}B_{kj} = ((\alpha A)B)_{ij}$.

(4) $(I_m A)_{ij} = \sum_{k=1}^{m}(I_m)_{ik}A_{kj} \underset{\text{only } k=i \text{ survives}}{=} 1\cdot A_{ij} = A_{ij}$

$(A\cdot I_n)_{ij} = \sum_{k=1}^{n} A_{ik}(I_n)_{kj} \underset{\text{only } k=j \text{ survives}}{=} A_{ij}\cdot 1 = A_{ij}$

All matrices have the same sizes & same entries, so the equalities in all 4 statements hold. $\square$

- **Next**: Relate addition, multiplication & scalar multiplication.

<u>Theorem 3</u>: (1) A & B of size $m \times n$ & C of size $n \times p$, then

$$(A+B)\,C = AC + BC \qquad (m \times p) \qquad \text{[Distribution 1]}$$

(2) A of size $m \times n$, B & C of size $n \times p$, then

$$A\,(B+C) = AB + AC \qquad (m \times p) \qquad \text{[Distribution 2]}$$

$(m \times n)$ (3) α, β scalars, A of size $m \times n$, then $(\alpha+\beta)A = \alpha A + \beta A$.

$(m \times n)$ (4) α scalar, A & B of size $m \times n$, then $\alpha(A+B) = \alpha A + \alpha B$.

<u>Proof</u>: Same idea as with Theorem 2. Check matrices on each side of $=$ have the same size & the same entries.

(1) $$((A+B)\,C)_{ij} = \sum_{k=1}^{n} (A+B)_{ik} C_{kj} = \sum_{k=1}^{n} (A_{ik} + B_{ik})\, C_{kj} =$$
$$= \sum_{k=1}^{n} (A_{ik} C_{kj}) + (B_{ik} C_{kj}) \underset{\substack{\downarrow \\ \text{reorder} \\ \text{sum}}}{=} \underbrace{\sum_{k=1}^{n} (A_{ik} C_{kj})}_{(AC)_{ij}} + \underbrace{\sum_{k=1}^{n} B_{ik} C_{kj}}_{(BC)_{ij}}$$

(2) $$(A\,(B+C))_{ij} = \sum_{k=1}^{n} A_{ik} (B+C)_{kj} = \sum_{k=1}^{n} A_{ik} (B_{kj} + C_{kj})$$
$$= \sum_{k=1}^{n} (A_{ik} B_{kj}) + (A_{ik} C_{kj}) \underset{\substack{\downarrow \\ \text{reorder} \\ \text{sum}}}{=} \underbrace{\sum_{k=1}^{n} A_{ik} B_{kj}}_{(AB)_{ij}} + \underbrace{\sum_{k=1}^{n} A_{ik} C_{kj}}_{(AC)_{ij}}.$$

(3) $$((\alpha+\beta)\,A)_{ij} = (\alpha+\beta)\,A_{ij} = \alpha A_{ij} + \beta A_{ij} = (\alpha A)_{ij} + (\beta A)_{ij} = (\alpha A + \beta A)_{ij}.$$

(4) $$(\alpha\,(A+B))_{ij} = \alpha (A+B)_{ij} = \alpha (A_{ij} + B_{ij}) = \alpha A_{ij} + \alpha B_{ij} = (\alpha A)_{ij} + (\alpha B)_{ij} = (\alpha A + \alpha B)_{ij}$$

§2 <u>The transpose of a matrix</u>:

<u>Idea</u>: Swap rows & columns.

<u>Def</u>: Given A of size $m \times n$, the <u>transpose of A</u> is a matrix A^T of size $n \times m$ with $(A^T)_{ij} = A_{ji}$ for $1 \leq i \leq n$, $1 \leq j \leq m$

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$

<u>Def</u>: A is <u>symmetric</u> if $A^T = A$ (in particular $m = n$ so A is a <u>square matrix</u>)

Theorem 4: A, B of size $m\times n$, C of size $n\times p$:

(1) $(A+B)^T = A^T + B^T$ $(n\times m)$

(2) $(AC)^T = C^T A^T$ $(p\times m)$

(3) $(A^T)^T = A$ $(m\times n)$

Proof: As usual, matrices on both sides have the same size, so we only need to check they have the same entries.

(1) $(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} + (A^T)_{ij} + (B^T)_{ij} = (A^T+B^T)_{ij}$

(2) $(AC)^T_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{ik} C_{kj} = \sum_{k=1}^{n} C_{kj} A_{ik} = \sum_{k=1}^{n} (C^T)_{jk} (A^T)_{ki}$
$= (C^T A^T)_{ji}$ □

(3) $(A^T)^T_{ij} = (A^T)_{ji} = A_{ij}$.

Property: Assume A is an $n\times n$ matrix. Then AA^T is symmetric, of size $n\times n$

Proof: $(\underset{n\times n}{A}\,\underset{n\times n}{A^T})^T = (A^T)^T A^T \underset{\text{Thm 4}}{=} A A^T$ so it's symmetric. □

Note: v in $\mathbb{R}^n$ ⇝ v^T is a row vector = matrix of size $1\times n$

Then $\|v\| = \sqrt{v\cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{v^T v}$ $v^T = [v_1, \ldots, v_n]$

Ex: $v = \begin{bmatrix}1\\2\end{bmatrix}$ $v^T = [1\ 2]$ ⇝ $\|v\| = \sqrt{v^T.v} = \sqrt{1^2+2^2} = \sqrt{5}$ $v = \begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix}$.

§3 Inverses of matrices:

Def: A matrix A of size $n\times n$ is invertible if we can find a matrix B of size $n\times n$ satisfying: $\boxed{AB = BA = I_n}$

Example: I_n is invertible ($B = I_n$ works)

Properties: If B exists, it is unique. Call it A^{-1}.

Why? Imagine B & B' both satisfy $AB = BA = I_n$
$AB' = B'A = I_n$

Then $B = B(\underbrace{AB'}_{I_n}) \underset{\text{Assoc}}{=} (\underbrace{BA}_{=I_n})B' = B'$.

Why? If $A \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}b_1\\ \vdots \\ b_n\end{bmatrix}$ A invertible, then $x = A^{-1}\begin{bmatrix}b_1\\ \vdots \\ b_n\end{bmatrix}$ is the UNIQUE solution to this