Lecture IX: §1.7 Linear independence & nonsingular matrices

Last Time: • Uniqueness of inverses of $n\times n$ matrices (whenever they exist)

• $A\underline{x}=\underline{b}$ in $\mathbb{R}^n$ with A invertible $n\times n$ matrix has a unique solution $\underline{x}=A^{-1}\underline{b}$ for any $\underline{b}$ in $\mathbb{R}^n$.

§1. Non-singular matrices:

Def: An $(n\times n)$ matrix A is nonsingular whenever $A\underline{x}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ in $\mathbb{R}^n$ has a unique solution, namely the trivial one $\underline{0}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ in $\mathbb{R}^n$.

Otherwise, we say A is singular.

Example: $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ is non-singular

$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is singular (Solutions: $\begin{cases}x_1=0\\ x_2 \text{ any}\end{cases}$, so $\underline{x}=x_2\begin{bmatrix}0\\1\end{bmatrix}$)

Prop: A is non-singular if and only if all $A\underline{x}=\underline{b}$ are consistent, with unique solution (for any $\underline{b}$ in $\mathbb{R}^n$)

Proof: • Assume $A\underline{x}=\underline{b}$ always has a unique solution, no matter what $\underline{b}$ is. Take special case: $b=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$. Then $A\cdot\underline{x}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ has a unique solution, so A is nonsingular by definition.

• For the converse; $A\cdot\underline{x}=\underline{0}$ has a unique solution, means $\underset{n\times n}{A}\sim\underset{n\times n}{A'}$ REF then A' has no row of zeroes, so $\operatorname{rank}(A)=\operatorname{rank}(A')=n$ in REF. This forces $A'=I_n$. So $[A|\underline{b}]\sim[I_n|b']$ gives $\underline{b}'$ as the unique soln to $A\underline{x}=\underline{b}$ $\square$
$\overset{\|}{A^{-1}}$

Obs: Our algorithm for finding A^{-1} gives $[A|I_n]\sim[I_n|B]$ if A is nonsingular. Then A is invertible (as discussed last time).

Consequence: If A is nonsingular, we found A^{-1}, showing A is invertible.

Thm 1: A of size $n\times n$ is invertible if and only if A is nonsingular.

Lemma: Fix P & Q of size $n\times n$, & write $R=PQ$

If P or Q are singular, then so is R.

Proof: • First, assume Q is singular. Then, we can find $\underline{x}$ in $\mathbb{R}^n$, $\underline{x} \neq \underline{0}$ with $Q \cdot \underline{x} = \underline{0}$. In turn, $R\,\underline{x} = (PQ)\underline{x} \underset{\text{Assoc}}{=} P(\underbrace{Q\,\underline{x}}_{=\underline{0}}) = \underline{0}$

So $R\,\underline{x} = \underline{0}$ admits a non-trivial solution. This shows R is singular.

• Now, assume Q is non-singular, so P must be singular. In particular we can find $\underline{b} \neq \underline{0}$ in $\mathbb{R}^n$ with $P \cdot \underline{b} = \underline{0}$ in $\mathbb{R}^n$.

Now: $Q \cdot \underline{x} = \underline{b}$ & Q is non-singular so by the Proposition this system has a unique solution, & $\underline{x} \neq \underline{0}$ because $\underline{b} \neq \underline{0}$.

But then $R\,\underset{\neq \underline{0}}{\underline{x}} = (PQ)\underline{x} \underset{\text{Assoc}}{=} P(Q\,\underline{x}) = P \cdot \underline{b} \underset{\text{def of } \underline{b}}{=} \underline{0}$ in $\mathbb{R}^n$.

Conclusion: R is singular.

Consequence: Can use the Lemma to show: "If $AB = I_n$ A, B of size $n \times n$ then $BA = I_n$ as well".

How? Take $P = A$, $Q = B$, $R = I_n$ in the Lemma. Since I_n is nonsingular by def, the Lemma forces A, B to be nonsingular.

In particular $B\,X = I_n$ admits a unique solution $X = C$ (break it into n systems)

Then $AB = BC = I_n$. As in Lecture 8, this forces $A = C$ so $B = A^{-1}$.

$[A = A\underbrace{(BC)}_{=I_n} = \underbrace{(AB)}_{=I_n}C = C]$ $\square$

§2. Linear independence in $\mathbb{R}^n$

Def: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is <u>linearly independent</u> (l.i.) if the only solution $(a_1, \ldots, a_p)$ to the vector equation

$$(*)\ \boxed{a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p = \underline{0} \text{ in } \mathbb{R}^n}$$

is the trivial one: $a_1 = a_2 = \cdots = a_p = 0$.

If a non-trivial solution to $(*)$ exists, we say $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is <u>linearly dependent</u> (there is an equation showing the vectors are "related").

• Start with examples! We'll see this notion leads to non-singular matrices

Ex 0 Any $\{\underline{0}, \vec{v}_2, \ldots, \vec{v}_p\}$ is linearly dependent since $1 \cdot \underline{0} + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = \underline{0}$

Example 1: $v_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix}$, $v_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

- $\{v_1, v_2, v_3\}$ are l.d. :

$$-2\,\vec{v_1} + (-1)\,\vec{v_2} + \vec{v_3} = \begin{bmatrix} -2\cdot 2 - 4 + 8 \\ -6 - 5 + 11 \\ -2 - 6 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $a_1 = -2$, $a_2 = -1$, $a_3 = 1$ is a nontrivial soln to $a_1 \vec{v_1} + a_2 \vec{v_2} + a_3 \vec{v_3} = \underline{0}$ in $\mathbb{R}^3$.

Q: How to solve the system in a_1, a_2, a_3?

$$(\text{LHS}) = \begin{array}{c} \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \\ \quad v_1 \;\; v_2 \;\; v_3 \end{array} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightsquigarrow$ linear system we can solve by Gauss-Jordan elimination

$$\begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 8 \\ 3 & 5 & 11 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & -13 & -13 \\ 0 & -8 & -8 \end{bmatrix} \xrightarrow[R_2 \to R_2/-13]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & -8 & -8 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 8R_1]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 6R_2]{} \underset{\text{REF}}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

$a_1 = -2a_3$
$a_2 = -a_3$
a_3 ANY

$\underline{a} = a_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

Special case: $a_3 = 1$
so $a_1 = -2$, $a_2 = -1$

- $\{v_1, v_2, v_4\}$ are l.i. :

System $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 2 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Solutions = ?

$$\begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 3 & 5 & 2 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -13 & 2 \\ 0 & -8 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3/-8]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -13 & 2 \\ 0 & 1 & -1/8 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 13R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 3/8 \\ 0 & 1 & -1/8 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -1/8 \\ 0 & 0 & 3/8 \end{bmatrix} \xrightarrow[R_3 \to \frac{8}{3} R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -1/8 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + \frac{1}{8} R_3 \\ R_1 \to R_1 - 6R_2}]{} I_3$$

So $a_1 = a_2 = a_3$ is the unique solution.

Q: Why study independence? Take $A \cdot \underline{x} = \underline{0}$ in $\mathbb{R}^n$ with A of size $n \times p$.

Prop 1: The system $A\,\underline{x} = \underline{0}$ has a unique solution if and only if, the p columns of A are linearly independent.

Why? $A\,\underline{x} = x_1\, \text{col}_1(A) + x_2\, \text{col}_2(A) + \cdots + x_p\, \text{col}_p(A) = \underline{0}$ in $\mathbb{R}^n$

Prop 2: The system $A\cdot\underline{x}=\underline{b}$ for $\underline{b}$ in $\mathbb{R}^n$, A of size $n\times p$ admits a solution if and only if $\underline{b}$ is a linear combination of the p columns A, meaning $\underline{b} = x_1\,\mathrm{col}_1(A) + x_2\,\mathrm{col}_2(A) + \cdots + x_p\,\mathrm{col}_p(A)$ for a choice of scalars $x_1,\ldots,x_p$.

The solution is unique only when the columns are linearly indep.

Theorem: If $\{\vec{v}_1,\ldots,\vec{v}_p\}$ are l.d. with $a_i\neq 0$, then $\vec{v}_i$ is a linear combination of the remaining $(p-1)$ vectors.

Why? $a_1\vec{v}_1+\cdots+a_i\vec{v}_i+\cdots+a_p\vec{v}_p = \mathbb{0}$, then

$$a_i\vec{v}_i = -\sum_{j\neq i} a_j\vec{v}_j \quad \text{in } \mathbb{R}^n$$

Since $a_i\neq 0$, multiply both sides by $\frac{1}{a_i}$ to get

$$\vec{v}_i = -\sum_{j\neq i} \underbrace{\frac{a_j}{a_i}}_{\text{in } \mathbb{R}}\vec{v}_j \,.$$

§3 Unit Vectors:

On $\mathbb{R}^n$ we have n unit vectors (analog of $\vec{i},\vec{j},\vec{k}$ in $\mathbb{R}^3$).

$$e_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\quad e_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\quad\ldots,\quad e_n=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

Prop 1: $\{e_1,\ldots,e_n\}$ are linearly independent:

Why? $\mathbb{0}\overset{?}{=}x_1e_1+\cdots+x_ne_n = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ gives $x_1=x_2=\cdots=x_n=0$.

Prop 2: Any vector $\underline{x}$ in $\mathbb{R}^n$ is a linear combination of $e_1,\ldots,e_n$:

Why? $\underline{x} = x_1e_1+\cdots+x_ne_n$.

§4 Useful properties:

Prop 3: Reordering a set of vectors preserves linear independence

Prop 4: A subset of a linearly independent set is also l.i.

Why? Say $\{\vec{v}_1,\ldots,\vec{v}_p\}$ is l.i. & pick $\{\vec{v}_1,\ldots,\vec{v}_s\}$.

If $a_1\vec{v}_1+\cdots+a_s\vec{v}_s=\mathbb{0}$ in $\mathbb{R}^n$ has a non-trivial soln $(a_1,\ldots,a_s)$, then $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \mathbb{0}$ has a solution $(\underbrace{a_1, \ldots, a_s}_{\neq \mathbb{0}}, 0, \ldots, 0)$

So $\{\vec{v}_1, \ldots, \vec{v}_p\}$ can't be linearly indepent!

Thm 2: If $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ & $p > n$, then the set is NEVER l.i.

Proof: We want to solve $\underbrace{[v_1 \cdots v_p]}_{A \text{ of size } n \times p} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$

eqns $= n <$ # unknowns $= p$, so we know the system has more than one solution! (infinitely many!) In particular, the vectors are l.d. $\square$.

Obs: If $p \leq n$, anything can happen:

(1) Unit vectors (and subsets of them) are l.i.

(2) any set containing $\mathbb{0}$ in $\mathbb{R}^n$ is l.d.

Thm 3: An $n \times n$ matrix A is nonsingular ($A \cdot \underline{x} = \mathbb{0}$ has unique soln $\underline{x} = \mathbb{0}$) if and only if the n columns are l.i.

Summary of results:

For A of size $n \times n$, the following statements are equivalent

(1) A is nonsingular

(2) The n columns of A are linearly independent

(3) $A\underline{x} = \underline{b}$ has ALWAYS a unique solution (for any choice of $\underline{b}$ in $\mathbb{R}^n$)

(4) A is invertible

(5) A is row equivalent to I_n