Lecture X: §2.1 Vectors in the Plane, §2.2 Vectors in Space

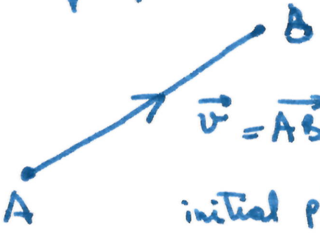
Recall: Vectors in $\mathbb{R}^n$ = $n \times 1$ matrices with entries in $\mathbb{R}$

TODAY: Geometric representation of vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$ (To model Physics, eg force, displacement, velocity ....)

§1. Vectors:

Def: Informally, a vector represents the data of magnitude (or length) & direction. Represent them by a directed line segment

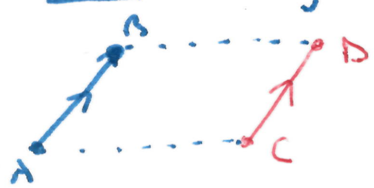
- arrow: indicated the direction
- length of segment = magnitude = $|\vec{v}|$



initial pt = A (Tail)
Terminal pt = B (head)

Observations:

① We don't distinguish between parallel vectors of the same magnitude.



$\overrightarrow{AB}$ & $\overrightarrow{CD}$ represent the same vector.

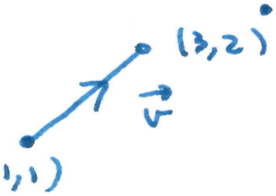
② $\vec{0}$ is the unique vector of magnitude 0 & no direction.

§2. Position Vectors & components:

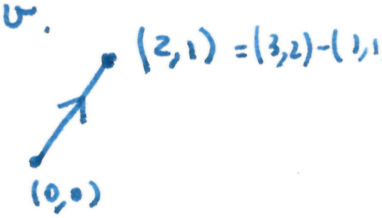
To have a consistent & algebraic representation of vectors, we ALWAYS fix the initial pt of $\vec{v}$ to be the origin O. This unique representative is called the position vector for $\vec{v}$.

Def head $(\vec{v}) = P$ ⤳ • position vector for $\vec{v} = \overrightarrow{OP}$
• coordinates of P = components of $\vec{v}$.

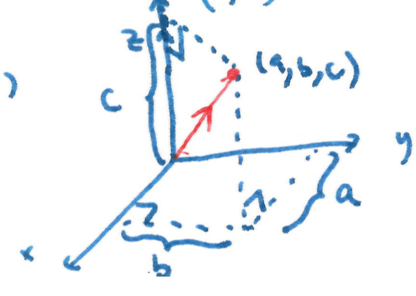
Examples (1)



⤳ position vector for $\vec{v}$



(2)



$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(3 right angles)

Ex $\mathbb{R}^2$

$B=(b_1,b_2)$, $\vec{v}$, $A=(a_1,a_2)$ $\rightsquigarrow$ x-comp of $\vec{v} = b_1-a_1$

y- ——— $= b_2 - a_2$

$\begin{bmatrix} b_1-a_1 \\ b_2-a_2 \end{bmatrix}$ position vector for $\vec{v}$.

$\mathbb{R}^3$:

$B=(b_1,b_2,b_3)$, $\vec{v}$, $A=(a_1,a_2,a_3)$ $\rightsquigarrow$ 3 components

x-comp $= b_1-a_1$

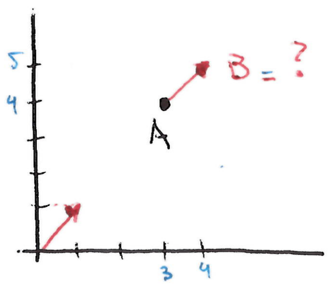
y- ——— $= b_2-a_2$

z- ——— $= b_3-a_3$

$\rightsquigarrow$ $\begin{bmatrix} b_1-a_1 \\ b_2-a_2 \\ b_3-a_3 \end{bmatrix}$ position vector for $\vec{v}$

Why? Equality of position vectors is determined by their components.

Example: given $v=\overrightarrow{AB}$ with position vector $\begin{bmatrix}1\\1\end{bmatrix}$. Find the coordinates of B given $A=(3,4)$ & draw $\vec{v}$



$B=(b_1,b_2)$
$A=(3,4)$

$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} b_1-3 \\ b_2-4 \end{bmatrix}$ means $1=b_1-3$, $1=b_2-4$

so $B=(4,5)$

Def: Magnitude $= \|\vec{v}\| = \begin{cases} \sqrt{v_1^2+v_2^2+v_3^2} & \text{in } \mathbb{R}^3 \quad \vec{v}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \\ \sqrt{v_1^2+v_2^2} & \text{in } \mathbb{R}^2 \quad v=\begin{bmatrix}v_1\\v_2\end{bmatrix} \end{cases}$

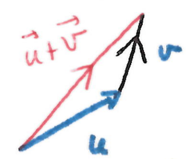
§3. Operations for vectors: Addition, scalar multiplication

- Algebraically = easy (vectors are $n\times 1$ matrices!)
- Geometrically = ? Draw in $\mathbb{R}^2$, but idea for $\mathbb{R}^3$ is the same

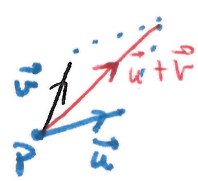
(1) **Addition**:

$\vec{u} + \vec{v} = ?$ Two ways

① **Triangle Law** = Put tail($\vec{v}$) at the head of $\vec{u}$ & draw $\triangle$.



② **Parallelogram Law**: Put 2 tails at the same point P, draw parallelogram & the diagonal from P to opposite corner



(2) Scalar multiplication:

$$\left.\begin{array}{l}\vec v \text{ vector}\\ c \text{ scalar}\end{array}\right\} \rightsquigarrow c\vec v \text{ vector with }\begin{array}{l}\cdot\ \text{magnitude} = |c|\,\|\vec v\|\\ \cdot\ \text{direction} = \begin{cases}\text{none} & \text{if } c=0\\ \text{same as } \vec v & \text{if } c>0\\ \text{opposite to } \vec v & \text{if } c<0\end{cases}\end{array}$$

Observe: ① $\vec v$ & $c\vec v$ are parallel vectors

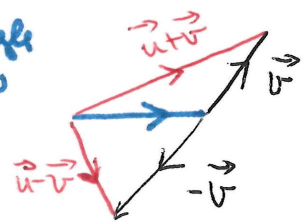
Eg:

② $\vec v \neq \vec 0$, then $\dfrac{\vec v}{\|\vec v\|}$ has magnitude one.

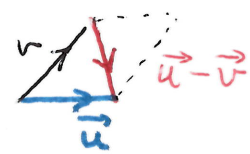
(3) Substraction / Difference: Combine (1) & (2)

$$\vec u - \vec v = \vec u + (-\vec v)$$

Triangle Law



Parallelogram Law: Take the other diagonal! (from $\vec v$ to $\vec u$)

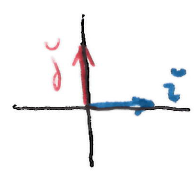


§4. Basic Unit Vectors:

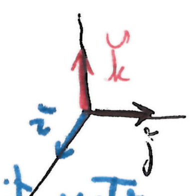
Def: Unit Vector = any vector of magnitude 1.

Examples ① $e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, …, $e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$ Basic unit Vectors

② [Calculus III] $\mathbb{R}^2$: $\check{i} = \begin{bmatrix}1\\0\end{bmatrix}$, $\check{j} = \begin{bmatrix}0\\1\end{bmatrix}$



$\mathbb{R}^3$: $\check{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\check{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\check{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$



Prop: Any vector in $\mathbb{R}^n$ is a linear combination of basic unit vectors

Eg: $\begin{bmatrix}a\\b\end{bmatrix} = a e_1 + b e_2$, $\begin{bmatrix}a\\b\\c\end{bmatrix} = a e_1 + b e_2 + c e_3$.

Ex: Find all vectors of length 8 parallel to $\vec v = \begin{bmatrix}3\\4\end{bmatrix}$

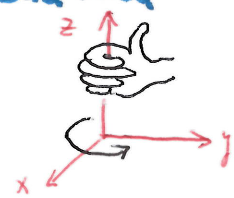
Sol: 1. Find 2 unit vectors parallel to $\vec v$: $\pm\dfrac{\vec v}{\|\vec v\|}$ (5, magnitude 1)

$$\frac{\begin{bmatrix}3\\4\end{bmatrix}}{5} = \begin{bmatrix}3/5\\4/5\end{bmatrix}, \qquad \frac{-\begin{bmatrix}3\\4\end{bmatrix}}{5} = \begin{bmatrix}-3/5\\-4/5\end{bmatrix}$$

2. Scale by 8 $\rightsquigarrow$ $\vec w = \pm\begin{bmatrix}24/5\\32/5\end{bmatrix}$

§5 Rectangular coordinates in $\mathbb{R}^3$:

- The 3 coordinate axes are directed according the the right hand rule



- 3 coordinate planes:

Plane	which axes?	equation
XY-plane	x- & y-	$z=0$
XZ-plane	x- & z-	$y=0$
YZ-plane	y- & z-	$x=0$

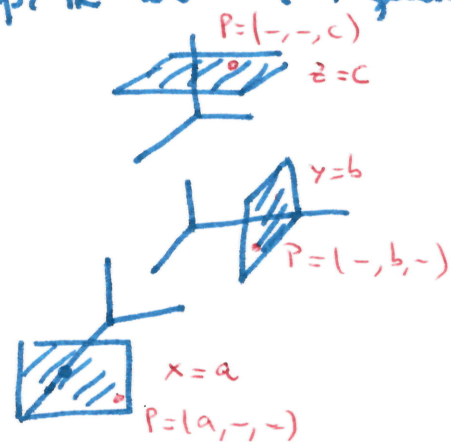
They subdivide $\mathbb{R}^3$ into <u>8 octants</u> (for $\mathbb{R}^2$ we had 4 quadrants)

Parallel planes to them: (1) $z=c$

⇓ (2) $y=b$

Translate plane so that (3) $x=a$

it passes through (a,b,c)



- Distances in $\mathbb{R}^3$: $d(P,Q)=\|\overrightarrow{PQ}\|$

- Special case: Midpoint M between P & Q.

$$P=(p_1,p_2,p_3) \quad Q=(q_1,q_2,q_3) \leadsto M=\left(\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}, \frac{p_3+q_3}{2}\right)$$

Why? M lies on the segment joining P & Q & $d(P,M)=d(M,Q)$

$$\overrightarrow{PM}=\begin{bmatrix}\frac{q_1-p_1}{2}\\ \frac{q_2-p_2}{2}\\ \frac{q_3-p_3}{2}\end{bmatrix}, \quad \overrightarrow{PQ}=\begin{bmatrix}q_1-p_1\\ q_2-p_2\\ q_3-p_3\end{bmatrix}, \quad \overrightarrow{MQ}=\begin{bmatrix}\frac{q_1-p_1}{2}\\ \frac{q_2-p_2}{2}\\ \frac{q_3-p_3}{2}\end{bmatrix}$$

so $\overrightarrow{PM}=\overrightarrow{MQ}=$
$=\frac{1}{2}\overrightarrow{PQ}$