Lecture XII: §2.3 Cross product & §2.4 Lines & planes

§1 Cross product in $\mathbb{R}^3$:

$\vec{u}, \vec{v}$ in $\mathbb{R}^3$ $\rightsquigarrow$ $\vec{u}\times\vec{v}$ is a vector in $\mathbb{R}^3$ $= \det\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$

<u>Example</u>: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{u}\times\vec{v} = \det\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = i\,(2^2 - (-1)3) - (1\cdot 2 - 2\cdot 3)\vec{j} + (-1-2^2)\vec{k}$

$= 7\vec{i} - (-4)\vec{j} + (-5)\vec{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$

<u>Properties</u>: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, α, β scalars

(1) $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ [<u>ANTI</u>COMMUTATIVE], so $\vec{u}\times\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$.

(2) $\alpha\vec{u}\times\beta\vec{v} = (\alpha\beta)(\vec{u}\times\vec{v})$ [ASSOCIATIVE], so $\vec{0}\times\vec{u} = \vec{0}$ for all $\vec{u}$

(3) $\left.\begin{aligned} \vec{u}\times(\vec{v}+\vec{w}) &= \vec{u}\times\vec{v} + \vec{u}\times\vec{w} \\ (\vec{u}+\vec{v})\times\vec{w} &= \vec{u}\times\vec{w} + \vec{v}\times\vec{w} \end{aligned}\right\}$ [DISTRIBUTIVE]

(4) $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

<u>Proof</u>: (1), (2) & (3) follow from properties of 2x2 determinants:

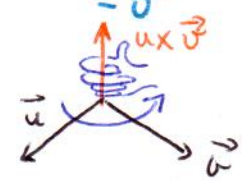
- $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -\det\begin{vmatrix} c & d \\ a & b \end{vmatrix}$
- $\det\begin{vmatrix} \alpha a & \alpha b \\ \beta c & \beta d \end{vmatrix} = (\alpha a)(\beta d) - (\alpha b)(\beta c) = \alpha\beta(ad-bc) = \alpha\beta\det\begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- $\det\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = (a+a')d - (b+b')c = (ad-bc) + (a'd - b'c) = \det\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \det\begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

(4) is a direct computation.

- <u>Key Property</u>: $\vec{u}\times\vec{v} \perp \vec{u}$ & $\vec{u}\times\vec{v} \perp \vec{v}$.

<u>Why?</u> Use dot product
$\vec{u}\cdot(\vec{u}\times\vec{v}) = (\vec{u}\times\vec{u})\cdot\vec{v} = \vec{0}\cdot\vec{v} = 0$
$\vec{v}\cdot(\vec{u}\times\vec{v}) = (-1)\,\vec{v}\cdot(\vec{v}\times\vec{u}) = -\underbrace{(\vec{v}\times\vec{v})}_{=\vec{0}}\cdot\vec{u} = 0.$

- Direction of $\vec{u}\times\vec{v}$ is given by the <u>right-hand rule</u> (or has no direction if $\vec{u}\times\vec{v} = \vec{0}$)



- <u>Geometry of cross products</u>: If $\vec{u}, \vec{v} \neq \vec{0}$, fix θ = angle between $\vec{u}$ & $\vec{v}$.

Then $\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\,\sin\theta}$

Proof: Write $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Verify that $\|\vec{u} \times \vec{v}\|^2 + (\vec{u} \cdot \vec{v})^2 = \|\vec{u}\|^2 \|\vec{v}\|^2$ (simple computation)

Then, since $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, we get

$$\|\vec{u} \times \vec{v}\|^2 + \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta = \|\vec{u}\|^2 \|\vec{v}\|^2$$

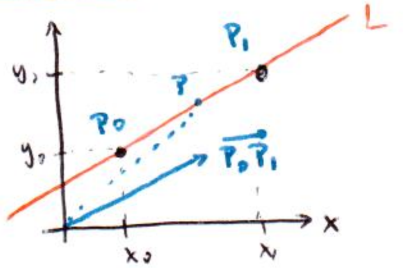
$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta) = \|\vec{u}\|^2 \|\vec{v}\|^2 \operatorname{sen}^2\theta \qquad (*)$$

Since $0 \leq \sin\theta$ for $0 \leq \theta \leq 180°$, we have $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \operatorname{sen}\theta$ by taking $\sqrt{\ }$ on both sides of $(*)$

Conclusion: (1) We know the magnitude & direction of $\vec{u} \times \vec{v}$

(2) $\vec{u} \times \vec{v} = \vec{0}$ for $\vec{u}, \vec{v} \neq \vec{0}$ if and only if $\theta = 0$ or $180°$. ($\vec{u} \,/\!/\, \vec{v}$)

§2. Lines in $\mathbb{R}^2$ & $\mathbb{R}^3$:



$P_0 = (x_0, y_0)$
$P_1 = (x_1, y_1)$

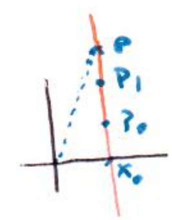
- Assume $x_0 \neq x_1$

  The equation of the line L through P_0 & P_1 is

  $$\boxed{y = m(x - x_0) + y_0}$$

  with $m = \text{slope} = \dfrac{y_1 - y_0}{x_1 - x_0}$

- If $x_0 = x_1$, the line is vertical & has equation: $\boxed{x = x_0}$

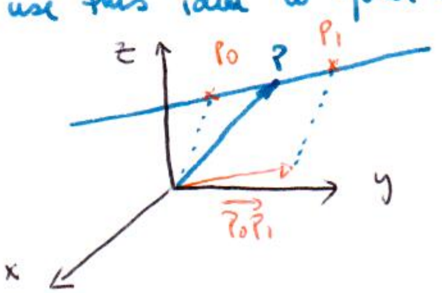


Parametric or vector form: ?

- direction of L : $\overrightarrow{P_0P_1}$

$\rightsquigarrow$ $\boxed{\overrightarrow{OP} = t\,\overrightarrow{P_0P_1} + \overrightarrow{OP_0} \quad \text{for some } t \text{ in } \mathbb{R}}$ (works for any slope!)

Can use this idea to find equations describing lines in $\mathbb{R}^3$.



- Two points (P_0, P_1) in $\mathbb{R}^3$ uniquely determine a line
- Vector Eqn: $\overrightarrow{OP} = t\,\overrightarrow{P_0P_1} + \overrightarrow{OP_0}$ for t in $\mathbb{R}$
- P lies in L if and only if $\overrightarrow{PP_0}$ is parallel to $\overrightarrow{P_0P_1} = \overrightarrow{OP} - \overrightarrow{OP_0}$ (if parallel, then $= t\,\overrightarrow{P_0P_1}$)

Parametric equation = ? Use the components of the vectors

Write $P_0 = (x_0, y_0, z_0)$ & $\vec{u}$ = direction of $L = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

If $P = (x, y, z)$, then P lies in L if and only if $\overrightarrow{P_0P} = t\vec{u}$ for some t.

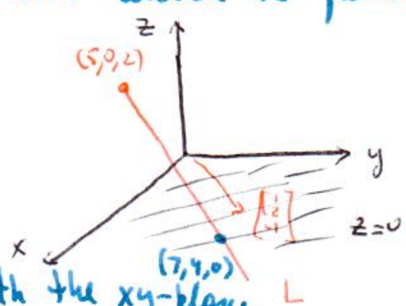
In coordinates: $\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

so $\begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases}$ for some t $\longleftrightarrow$ $\begin{cases} x = \boxed{x_0} + \boxed{a}\,t \\ y = \boxed{y_0} + \boxed{b}\,t \\ z = \boxed{z_0} + \boxed{c}\,t \end{cases}$ for some t

point P_0 ; direction of L

<u>Example</u>: Find the equation of the line which is parallel to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & passes through $(5, 0, 2)$

<u>A</u>: $\begin{cases} x = 5 + t \\ y = 0 + 2t \\ z = 2 + (-t) \end{cases}$



Find the intersection of this line with the xy-plane.

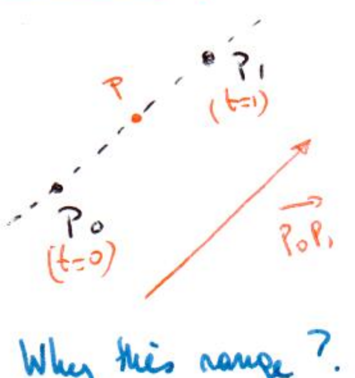
. xy-plane : $z = 0$

$\rightsquigarrow$ $2 - t = 0$ so $\boxed{t = 2}$

Substitute t in other 2 components $\quad x = 5 + 2 = 7$, $\quad y = 0 + 2 \cdot 2 = 4$

<u>A</u>: $Pt = (7, 4, 0)$

§3 <u>Line Segments</u>:



We restrict the equation of the line through P_0 & P_1 to <u>specific</u> values of t

$$\boxed{\overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1} \quad \text{for } 0 \leq t \leq 1}$$

Why this range ? $\quad$. $P = P_0$ for $t = 0$ $\quad (0 = \overrightarrow{P_0P_0} = 0 \cdot \overrightarrow{P_0P_1})$

. $P = P_1$ for $t = 1$ $\quad (\overrightarrow{P_0P_1} = 1 \cdot \overrightarrow{P_0P_1})$

So points in between correspond to values of t in between 0 & 1.

<u>Special case</u> : $t = \frac{1}{2}$ $\rightsquigarrow$ midpoint between P_0 & P_1 (last time)

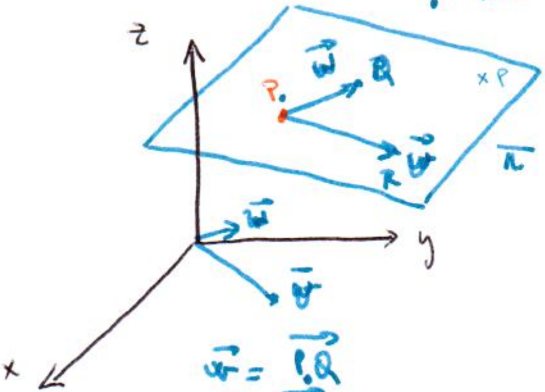
$P_0 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $P_1 = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ $\quad$ midpoint $= \begin{bmatrix} \frac{a_1 + b_1}{2} \\ \vdots \\ \frac{a_n + b_n}{2} \end{bmatrix}$ $\rightsquigarrow \|\overrightarrow{P_0P}\| = \frac{1}{2}\|\overrightarrow{P_0P_1}\|$

§2. Planes in 3-Spaces

Two ways to determine a plane in 3-space

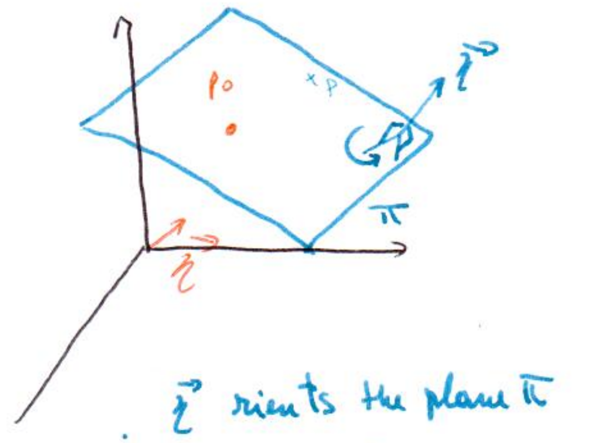
① A point P_0 & 2 non-parallel directions $(\vec{v}, \vec{w})$

[Equivalently : 3 non-collinear pts P_0, Q, R]



$\vec{w} = \overrightarrow{P_0Q}$
$\vec{v} = \overrightarrow{P_0R}$

② A point P_0 & a normal $\vec{n}$



- $\vec{n}$ orients the plane π
- $\vec{n}$ is normal to π if $\vec{n}$ is perpendicular to the 2 directions of π

$$\boxed{\vec{n} = \vec{v} \times \vec{w}}$$

($\underbrace{\vec{w} \times \vec{v}}_{=-\vec{n}}$ also works)

We know $\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{w} = 0$

- Vector equation for π : $\overrightarrow{P_0P} \cdot \vec{n} = 0$

Explicitly: $\left.\begin{array}{l} P = (x, y, z) \\ P_0 = (x_0, y_0, z_0) \\ \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0} \end{array}\right\}$ $\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{fixed \#.}}$$

- Conversely, from the equation we get $\vec{n}$ coefficients of the
  - any explicit solution gives P_0.

Example : Find the equation of the plane passing through $P_0 = (1,0,0)$, $R_0 = (1,1,1)$, $Q_0 = (2,1,-1)$.
Compute the intersection of this plane with the 3 coordinate planes (xy-, xz- & yz-planes)