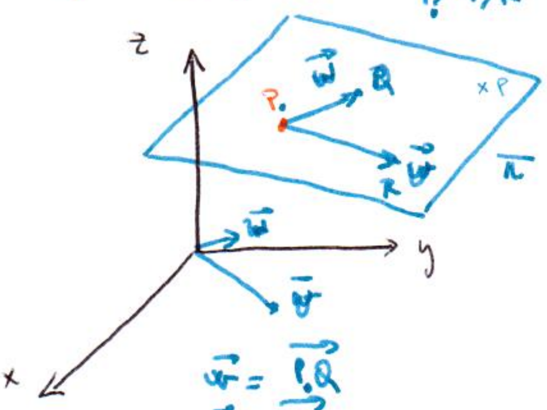
§2. Planes in 3-spaces

Two ways to determine a plane in 3-space

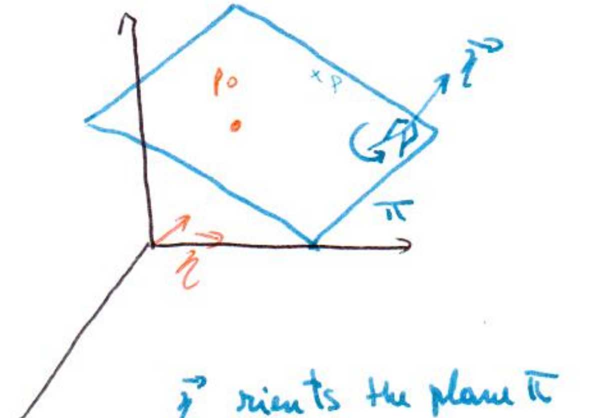
① A point P_0 & 2 non-parallel directions $(\vec{v}, \vec{w})$

[Equivalently: 3 non-collinear pts P_0, Q, R]



$\vec{w} = \overrightarrow{P_0Q}$
$\vec{v} = \overrightarrow{P_0R}$

② A point P_0 & a normal $\vec{n}$



. $\vec{n}$ orients the plane π

. $\vec{n}$ is normal to π if $\vec{n}$ is perpendicular to the 2 directions of π

$$\boxed{\vec{n} = \vec{v} \times \vec{w}}$$

($\vec{w} \times \vec{v}$ also works) $= -\vec{n}$

We know $\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{w} = 0$

. Vector equation for π : $\overrightarrow{P_0P} \cdot \vec{n} = 0$

Explicitly: $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$, $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{fixed \#.}}$$

. Conversely, . from the coefficients of the equation we get $\vec{n}$

. any explicit solution gives P_0.

Example: Find the equation of the plane passing through $P_0 = (1,0,0)$, $R_0 = (1,1,1)$, $Q_0 = (2,1,-1)$.

Compute the intersection of this plane with the 3 coordinate planes (xy-, xz- & yz-planes)

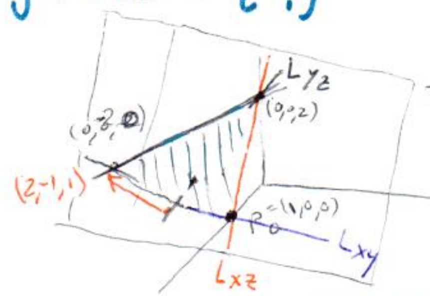
2 directions ; $v = P_0Q_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{w} = \overrightarrow{P_0R_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

So $\vec{\eta} = \vec{v} \times \vec{w} = \det \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i}\,(1-(-1)) - \vec{j}\cdot 1 + \vec{k} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Equation : $2(x-\underbrace{1}_{P_0}) + (-1)(y-0) + (z-0) = 0$

$$\boxed{2x - y + z = 2}$$



. Easy check : P_0, Q_0 & R_0 satisfy the equation

. The 3 intersections will be lines in the corresponding coordinate planes.

xy-plane : $\begin{cases} z = 0 \\ 2x - y + z = 2 \end{cases}$ line $L_{xy} = 2x - y = 2$ in xy-plane

yz-plane : $\begin{cases} x = 0 \\ 2x - y + z = 2 \end{cases}$ — $L_{yz} = z - y = 2$ — yz-plane

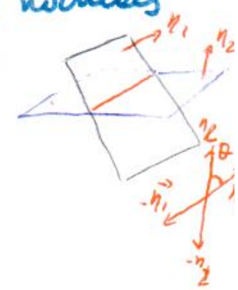
xz-plane $\begin{cases} y = 0 \\ 2x - y + z = 2 \end{cases}$ — $L_{xz} = 2x + z = 2$ — xz-plane

3 intersections between the lines $(0,0,2)$, $(0,-2,0)$ & $(1,0,0)$

3 Parallel and orthogonal planes:

Def : Angles between 2 planes = (acute) angle between their normals

(Use $|\vec{\eta}_1 \cdot \vec{\eta}_2| = \|\vec{\eta}_1\|\,\|\vec{\eta}_2\|\,|\cos\theta|$ to find θ with $0 \le \theta \le 90°$



In particular : $(\theta = 0)$ parallel planes : $\vec{n}_1 \parallel \vec{n}_2$

$(\theta = 90°)$ orthogonal planes : $\vec{n}_1 \perp \vec{n}_2$.

Example : Find the parallel plane to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

Soln $\vec{\eta} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ π : $3(x-1) + (-2)(y+1) + 5(z-1) = 0$

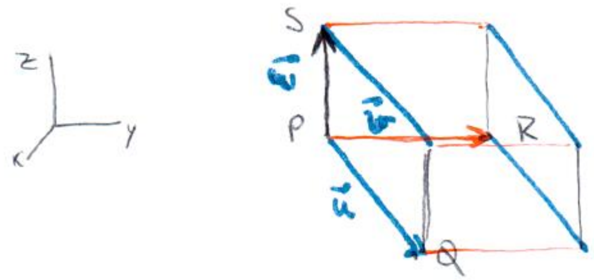
$$\boxed{3x - 2y + 5z = 10}$$

Example : Find a plane orthogonal to $3x - 2y + 5z = 4$, passing through $P_0 = (1,0,0)$ & $Q_0 = (1,1,0)$

Soln : normal $\vec{\eta} \perp \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

— $\vec{\eta} \perp \overrightarrow{P_0Q_0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Bigg\} \rightsquigarrow \vec{\eta} = \det \begin{vmatrix} i & j & k \\ 3 & -2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$ π : $\boxed{-5x + 3z = -5}$

3) <u>Coplanar points</u>: 4 points P, Q, R, S are coplanar if and only if the 3 vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ & $\overrightarrow{PS}$ form a <u>flat</u> <u>parallelepiped</u>. (ie Volume $=0$)



$$\text{Vol} = \|\vec{u}\cdot(\vec{v}\times\vec{w})\| = \|(\vec{u}\times\vec{v})\cdot\vec{w}\|$$

$\vec{u} = \overrightarrow{PS}$

$\vec{v} = \overrightarrow{PR}$

$\vec{w} = \overrightarrow{PS}$

<u>Example</u>: Show that $\underset{P}{(1,3,2)}$, $\underset{Q}{(3,-1,6)}$, $\underset{R}{(5,2,0)}$, $\underset{S}{(3,6,-4)}$ are coplanar

<u>Soln 1</u>: $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ $\leadsto$ $\text{Vol} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix} = 0$ ✓

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix}$$

<u>Soln 2</u>: Find the equation of the plane through a choice of 3 pts (P, R, S) & check 4th point also verifies the equation.

$\vec{n} = \vec{v}\times\vec{w} = \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix}$ $\quad\pi: 12x + 20y + 14z = 12\cdot 1 + 20\cdot 3 + 14\cdot 2 = 100$

Check for Q: $12\cdot 3 - 20\cdot 1 + 14\cdot 6 \overset{?}{=} 100$ ✓