

§1. The vector space \mathbb{R}^n & its properties:

- All Vectors in $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$
- Solutions to homogeneous systems in \mathbb{R}^n

$$\underline{x} = x_{i1} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} + x_{i2} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} + \dots + x_{ij} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

(lin comb. of vectors in \mathbb{R}^n)

Q: What properties do they have?

- $\vec{0}$ lies in the space
- add vectors / solutions & we remain a vector / solution
- scalar multiplication preserves \mathbb{R}^n / space of solutions

Note: ①-③ fail for non-homogeneous systems!

- These properties will characterize vector spaces (\mathbb{R}^n will be our favorite example)

Theorem 1: Write $W = \mathbb{R}^n$. For x, y, z in W , α, β scalars, we have the following properties:

- Closure properties: (C1) x, y in W , then $x+y$ in W
(②, ③ above) (C2) x in W , αx in W

- Addition properties: (A1) $x+y = y+x$ [COMMUTATIVE]
(A2) $x+(y+z) = (x+y)+z$ [ASSOCIATIVE]
(A3) $\vec{0}$ in W satisfies $x+\vec{0} = \vec{0}+x = x$ [NEUTRAL ELEMENT]
(A4) given x in W , we can find Y in W with
 $x+Y = \vec{0}$ ($Y = "-x" = (-1) \cdot x$) [ADDITIVE INVERSE]

- Scalar Multiplication properties: (M1) $\alpha(\beta x) = (\alpha\beta)x$ [ASSOCIATIVE]
(M2) $\alpha(x+y) = \alpha x + \alpha y$ [DISTRIBUTIVE I]
(M3) $(\alpha+\beta)x = \alpha x + \beta x$ [———— II]
(M4) $1 \cdot x = x$ for all x in W .

Note: (A4) follows from C2

Def: A subset W of \mathbb{R}^n satisfying these 10 properties is called a (vector) subspace of \mathbb{R}^n .

Note: $A_1, A_2, A_3, \dots, A_4$ are inherited from \mathbb{R}^n , so we only need to check C_1, C_2, A_3 & A_4 . But A_4 will follow from C_2 because $-x = (-1)x$.

Theorem 2: A subset \mathbb{V} of \mathbb{R}^n is a subspace if and only if;

- (S1) The zero vector $\vec{0}$ is in \mathbb{V}
- (S2) $x+y$ lies in \mathbb{V} whenever x, y are in \mathbb{V}
- (S3) αx is in \mathbb{V} & α is any scalar in \mathbb{R} .

Ex. 2. Examples / Non-examples:

Meta example: A solution set to a homogeneous system of m equations in n unknowns.

Write it as $\{ \underline{x} \in \mathbb{R}^n \mid A \cdot \underline{x} = \underline{0} \}$

- (S1) The Trivial soln is $\underline{0}$ ✓
- (S2) $A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$ ✓ $\Rightarrow \underline{x} + \underline{y}$ is a soln.
- (S3) $A(\alpha \underline{x}) = \alpha(A\underline{x}) = \alpha \cdot \underline{0} = \underline{0}$ ✓ $\Rightarrow \alpha \underline{x}$ is a soln.

Example 1: A line L in \mathbb{R}^3 through $(0,0,0)$, e.g. $\begin{cases} x+y+z=0 \\ x+2y-z=0 \end{cases}$

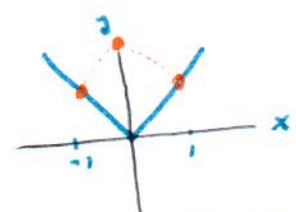
- (S1) $[0,0,0]^t$ in the line ✓
- (S2) $\begin{matrix} [x_1, y_1, z_1]^t \\ [x_2, y_2, z_2]^t \end{matrix}$ in $L \Rightarrow \begin{cases} (x_1+x_2) + (y_1+y_2) + (z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = 0+0=0 \\ (x_1+x_2) + 2(y_1+y_2) - (z_1+z_2) = (x_1+2y_1-z_1) + (x_2+2y_2-z_2) = 0+0=0 \end{cases}$

So $[x_1+x_2, y_1+y_2, z_1+z_2]^t$ in L

- (S3) α in \mathbb{R} $\left. \begin{matrix} [x_1, y_1, z_1]^t \text{ in } L \end{matrix} \right\} \Rightarrow \begin{cases} \alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha(x_1+y_1+z_1) = \alpha \cdot 0 = 0 \\ \alpha x_1 + 2(\alpha y_1) - \alpha z_1 = \alpha(x_1+2y_1-z_1) = \alpha \cdot 0 = 0 \end{cases}$

So $\alpha [x_1, y_1, z_1]^t$ in L .

Non-example: Graph of $|x|$ in \mathbb{R}



- (S1) holds
- (S2) fails $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \underline{z} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \underline{x} + \underline{z} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ not in the graph!

Non-example 2: $\mathbb{V} = \{ \underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 3 \end{bmatrix}, x_1, x_2 \in \mathbb{R} \}$ = plane with equation $z=3$

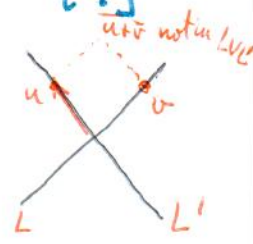
(S1) fails

Non-example 3: $W = \{x : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_1, x_2 \text{ integers}\}$

(S1) holds

(S2) — (sum of integers is an integer)

(S3) does NOT hold $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ not in W , but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is.



Non-example 4: Union of 2 different lines in \mathbb{R}^2 through (0,0)

§2. The span of a subset:

Def: Given vectors $\vec{v}_1, \dots, \vec{v}_r$ in \mathbb{R}^n , we write

$$W = Sp(\vec{v}_1, \dots, \vec{v}_r) = \text{set of all linear combinations } \vec{v}_1, \dots, \vec{v}_r$$

$$= \{ \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R} \}$$

Theorem 2: The set $W = Sp(\vec{v}_1, \dots, \vec{v}_r)$ is a subspace of \mathbb{R}^n

Proof: (S1) $\vec{0} = 0 \cdot \vec{v}_1 + \dots + 0 \cdot \vec{v}_r$ ✓

(S2) \vec{x} in W : $\vec{x} = a_1 \vec{v}_1 + \dots + a_r \vec{v}_r$ a₁, ..., a_r scalars
 \vec{y} " " $\vec{y} = b_1 \vec{v}_1 + \dots + b_r \vec{v}_r$ b₁, ..., b_r —

$$\vec{x} + \vec{y} = (a_1 + b_1) \vec{v}_1 + \dots + (a_r + b_r) \vec{v}_r$$

(a₁+b₁), ..., (a_r+b_r) scalars.

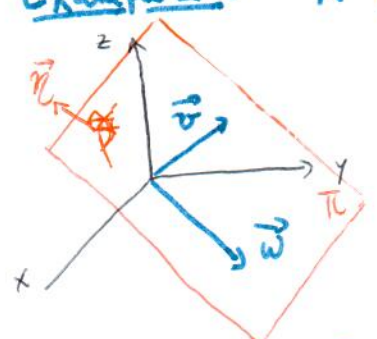
(S3) $\alpha \vec{x} = (\alpha a_1) \vec{v}_1 + \dots + (\alpha a_r) \vec{v}_r$ αa₁, ..., αa_r scalars.

Example 1: A line L through the origin in \mathbb{R}^n :

Vector equation $\vec{OP} = \vec{0} + t\vec{v} = t\vec{v}$ v = direction of L

So $L = Sp(\vec{v})$.

Example 2: A plane π in \mathbb{R}^3 passing through the origin.



Eqn $\vec{OP} \cdot \vec{z} = 0$ $\vec{v} \times \vec{w}$
 \vec{v}, \vec{w} non-collinear vectors $\vec{OP} \cdot \vec{z} = 0$ becomes
 $\vec{OP} = \alpha \vec{v} + \beta \vec{w}$

Conclude: $\pi = Sp(\vec{v}, \vec{w})$

Why? $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{z} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ Eqn: $-x + 2y - z = 0$

• \vec{y} in $Sp(\vec{v}, \vec{w})$ so $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2a+b \\ -a+b \\ b \end{bmatrix}$

• Given \vec{y} we must find a, b that solves the system of 3 equations in 2 unknowns

Augmented matrix $\left[\begin{array}{cc|c} -2 & 1 & y_1 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{array} \right] \xrightarrow{\substack{R_2 \leftrightarrow R_1 \\ R_1 \rightarrow -R_1}} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ -2 & 1 & y_1 \\ 0 & 1 & y_3 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & -1 & y_1 - 2y_2 \\ 0 & 1 & y_3 \end{array} \right]$

$\xrightarrow{\substack{R_3 \rightarrow R_3 + R_2 \\ R_2 \rightarrow -R_2}} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & 1 & -y_1 + 2y_2 \\ 0 & 0 & y_1 - 2y_2 + y_3 \end{array} \right]$

The system is consistent if and only if $y_1 - 2y_2 + y_3 = 0$ (= -Eqn of π).

§4. The Null Space of a Matrix:

Def: Given an $m \times n$ matrix A , the solution set to $A \cdot \underline{x} = \underline{0}$ in \mathbb{R}^n is the Null Space (or Kernel) of A :

$N(A) = \{x: Ax = 0, x \text{ in } \mathbb{R}^n\}$

Theorem 3: $N(A)$ is a subspace of \mathbb{R}^n .

Obscure: $N(A) = \left\{ x_1 \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} + \dots + x_r \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \right\} = Sp(\vec{v}_1, \dots, \vec{v}_r)$

Example $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 2 & 8 & 5 \\ -1 & -2 & -4 & 1 \end{bmatrix} \rightsquigarrow [A|0] \xrightarrow{\text{row equiv}} \begin{bmatrix} 1 & 0 & 2 & 3 & | & 0 \\ 0 & 1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

$\begin{cases} x_1 = -2x_3 - 3x_4 \\ x_2 = x_3 + 2x_4 \end{cases} \rightsquigarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow N(A) = Sp\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right)$

Example: A plane through the origin in \mathbb{R}^3 $\vec{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ Eqn $ax + by + cz = 0$ $N([a \ b \ c])$.