Lecture XV: §3.3 More examples of subspaces of $\mathbb{R}^n$
§3.4. Bases for Subspaces

Recall: A subset $\mathbb{V}$ of $\mathbb{R}^n$ is a vector subspace if it satisfies:

(S1) $\mathbb{0}$ in $\mathbb{V}$

(S2) if $\underline{x}$, $\underline{y}$ in $\mathbb{V}$, then $\underline{x}+\underline{y}$ is also in $\mathbb{V}$

(S3) if $\underline{x}$ in $\mathbb{V}$, α scalar, then $\alpha\underline{x}$ is also in $\mathbb{V}$.

Main examples ① $\mathbb{V} = Sp(\vec{v}_1, \ldots, \vec{v}_r) = \{\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \mid \alpha_1, \ldots \alpha_r \text{ in } \mathbb{R}\}$ (span of a subset)

② $\mathbb{V} = \{\underline{x} : A\underline{x} = \mathbb{0}\} = \mathcal{N}(A)$ $\quad A = m \times n$ matrix [Null space or kernel of A]

Q: More examples?

§1 The range of a matrix:

Def: Given an $m\times n$ matrix A, the range of A is the set of vectors:

$$\mathcal{R}(A) := \{ y \text{ in } \mathbb{R}^m : y = Ax \text{ for some } x \text{ in } \mathbb{R}^n \}$$

Obs: $Ax = x_1 \operatorname{col}_1(A) + x_2\operatorname{col}_2(A) + \cdots + x_n \operatorname{col}_n(A)$ $\quad x_1, \ldots x_n$ in $\mathbb{R}$

Conclude: $\mathcal{R}(A) = Sp(\text{columns of } A)$ = column space of A = Columns(A)

In particular, it is a subspace of $\mathbb{R}^m$.

Example: $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -3 \end{bmatrix}$

$\underline{y} = A\underline{x} \rightsquigarrow [A \mid y] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 2 & -1 & 4 & 0 & y_2 \\ 1 & 1 & 5 & -3 & y_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 2 & 4 & -4 & y_3 - y_1 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 2R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 0 & \boxed{3y_1 - 2y_2 + y_3} \end{array}\right]$

The system is compatible if and only if $\boxed{3y_1 - 2y_2 + y_3 = 0}$ = plane in $\mathbb{R}^3$

So $y_3 = -3y_1 + 2y_2$ $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ -3y_1 + 2y_2 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Conclude: $\mathcal{R}(A) = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \mathcal{N}([3\ -2\ 1])$

§2. The Row Space of a matrix:

Def: Given an $m\times n$ matrix A, we define the Row Space of A as the span of its m rows (vectors in $\mathbb{R}^n$ after transposing)

Obs: $\mathrm{Rows}(A) = \mathrm{Columns}(A^t)$

Example: $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \end{bmatrix}$ $\quad \mathrm{Columns}(A) = \mathrm{Sp}\left(\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\right)$ in $\mathbb{R}^2$.

$\mathrm{Rows}(A) = \mathrm{Sp}\left(\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}5\\7\\8\end{bmatrix}\right)$ in $\mathbb{R}^3$.

Q: What happens under elementary row operations? A: Same row space!

Theorem 1: If $A \underset{\substack{\text{row}\\\text{equiv.}}}{\sim} B$, then A & B have the same row space.

Proof (idea): Enough to check nothing changed under each of the 3 elementary row operations [$A \xrightarrow{\text{elim}} A_1 \xrightarrow{\text{elim}} \cdots \xrightarrow{\text{elim}} A_s = B$ then $\mathrm{Row}(A) = \mathrm{Row}(A_1) = \cdots = \mathrm{Row}(\underset{\substack{\parallel\\ B}}{A_s})$]

(E1) Swapping rows clearly preserves the row space.

(E2) Multiplying a row (say R_1) by a scalar $\alpha \neq 0$:

$$\mathrm{Sp}(\alpha R_1, R_2, \ldots, R_m) = \mathrm{Sp}(R_1, R_2, \ldots, R_m)$$

$$\frac{\beta_1}{\alpha}(\alpha R_1) + \beta_2 R_2 + \cdots + \beta_m R_m = \beta_1 R_1 + \beta_2 R_2 + \cdots + \beta_m R_m$$

(E3) Multiplying a row (say R_1) & adding the result to a row (say R_2)

$$\mathrm{Sp}(R_1, \alpha R_1 + R_2, R_3, \ldots, R_m) = \mathrm{Sp}(R_1, R_2, \ldots, R_m)$$

$$\beta_1 R_1 + \beta_2(\alpha R_1 + R_2) + \beta_3 R_3 + \cdots + \beta_m R_m = \underbrace{(\beta_1 + \beta_2\alpha)}_{=\beta_1' \text{ and } \boxed{\beta_1 = \beta_1' - \beta_2\alpha}} R_1 + \beta_2 R_2 + \cdots + \beta_m R_m$$

Q: Why is this important?

A: We can use row operations to find a better set of generators for $\mathrm{Row}(A)$, and in general for any $\mathrm{Sp}(v_1, \ldots, v_m)$ in $\mathbb{R}^n$.

$$A = \begin{bmatrix} v_1^t \\ \vdots \\ v_m^t \end{bmatrix} \underset{\text{row-equiv}}{\sim} B = \begin{bmatrix} w_1 \\ \vdots \\ w_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boxed{W = \mathrm{Row}(A) = \mathrm{Sp}(w_1^t, \ldots, w_r^t)}$$

Example: $v_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\3\\5\end{bmatrix}$, $v_3 = \begin{bmatrix}3\\5\\6\end{bmatrix}$, $v_4 = \begin{bmatrix}-1\\-1\\-4\end{bmatrix}$ (4 generators)

- Write A with rows $v_1^t, v_2^t, v_3^t, v_m^t$ $\qquad A = 4\times 4$ matrix

• Find $A \sim B$ = reduced echelon form

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + R_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \\ R_2 \to -R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

• Take nonzero rows of B, transpose them. We get a better set of generators for $\mathbb{W}$

New set $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \rightsquigarrow \mathbb{W} = Sp(v_1, \dots, v_4)$ (4 generators)

$= Sp\left(\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right)$ (2 ———)

<u>Q</u>: Can we do better?

<u>A</u>: No! 2 is the minimal number we need! (Later on: call it <u>dimension</u> of $\mathbb{W}$)

Why? $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ are linearly independent (cannot have $\mathbb{W} = Sp(\vec{v})$, otherwise $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} = a\vec{v}$, $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = b\vec{v}$ $\quad a \neq 0,\ b \neq 0$ forces $b\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} - a\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \vec{0}$)

<u>Advantage 2</u>: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ forces $a = x_1$, $b = x_2$ &

$x_3 = 7a + b(-3) = 7x_1 - 3x_2 \rightsquigarrow \mathbb{W} = \{7x_1 - 3x_2 - x_3 = 0\}$

$= \mathcal{N}([7 \ -3 \ -1])$.

<u>§3. Spanning Sets & Bases</u>:

<u>Def</u>: Fix $\mathbb{W}$ a subspace of $\mathbb{R}^n$ & $S = \{\vec{v}_1, \dots, \vec{v}_r\}$ a subset of $\mathbb{R}^n$. (finite)

• S is a <u>spanning set</u> for $\mathbb{W}$ (or S spans $\mathbb{W}$) if $\mathbb{W} = Sp(\{\vec{v}_1 \cdots \vec{v}_r\})$

• S is a <u>minimal spanning set</u> for $\mathbb{W}$ if:

(1) S spans $\mathbb{W}$

(2) For <u>all</u> $i = 1 \dots r$, $S \setminus \{\vec{v}_i\} = \{\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots \vec{v}_r\}$ does NOT span $\mathbb{W}$.

<u>Def</u>: A <u>basis</u> for $\mathbb{W}$ is a finite minimally spanning set for $\mathbb{W}$.

<u>Next Time</u>: A finite set B is a <u>basis</u> for $\mathbb{W}$ if

(B1) B spans $\mathbb{W}$

(B2) B is linearly independent.