Lecture XVII: §3.5 Dimension of Subspaces

§1 Dimension:

In the examples from L14, we saw that the bases constructed all have the same size

Def: The dimension of a subspace W of $\mathbb{R}^n$ is the number of vectors in a basis for W.

Obs: If $W = \{\mathbb{0}\}$, then W has no basis, so $\dim W = 0$.

Example: $\dim \mathbb{R}^n = n$ for any $n \geq 1$ ($\{e_1, \ldots, e_n\}$ is a basis)

Q: How do we know all basis for W have the same number of elements?

A: Comes from the following fact:

Theorem 1: Fix $W \neq \{\mathbb{0}\}$ a subspace of $\mathbb{R}^n$ & $S = \{\vec{w}_1, \ldots \vec{w}_p\}$ a spanning set for W.
Then, any set of $p+1$ or more elements in W is linearly dependent.

Consequence: B, B' two basis for W, then size(B) = size(B').

Why? If $\# B = p < \# B'$ then B' has $\geq p+1$ elements & so it must be l.d. by the theorem. This can't happen because B' is l.i.

Proof of Theorem 1: Pick $\vec{v}_1, \ldots \vec{v}_m$ in W with $m \geq p+1$. Write:

$$\begin{cases} \vec{v}_1 = a_{11}\vec{w}_1 + \cdots + a_{p1}\vec{w}_p \\ \vec{v}_2 = a_{12}\vec{w}_1 + \cdots + a_{p2}\vec{w}_p \\ \vdots \\ \vec{v}_m = a_{1m}\vec{w}_1 + \cdots + a_{pm}\vec{w}_p \end{cases}$$

(Can do this because $W = Sp(\vec{w}_1, \ldots \vec{w}_p)$)

GOAL: Find a nontrivial solution to $\alpha_1\vec{v}_1 + \cdots + \alpha_m\vec{v}_m = \mathbb{0}$.

Rewrite it as $\vec{\mathbb{0}} = \underbrace{[v_1 \cdots v_m]}_{\text{size } n\times m} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \underbrace{[\vec{w}_1 \cdots \vec{w}_p]}_{\substack{W \\ \text{size } n\times p}} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & & a_{pm} \end{bmatrix}}_{A \text{ size } p\times m} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$

(in $\mathbb{R}^p$)

Since $m > p$, columns of A are l.d. In particular, A is singular

We can find $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \neq \mathbb{0}$ with $A\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \mathbb{0}$.

Then $[v_1 \cdots v_m]\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = W A \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = W\,\mathbb{0} = \mathbb{0}$ & $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \neq \mathbb{0}$, so $\{\vec{v}_1, \ldots \vec{v}_m\}$ is l.d.

(Recall: A set of $m > p$ vectors in $\mathbb{R}^p$ is l.d.) $\square$

<u>Theorem 2</u>: Fix $\mathbb{V} \neq \{0\}$ a subspace of $\mathbb{R}^n$ with $\dim \mathbb{V} = p$. Then:

(1) A set of $p+1$ or more vectors in $\mathbb{V}$ is linearly dependent

(2) Any set of $p-1$ or fewer ——— can't span $\mathbb{V}$.

(3) ——— p linearly independent vectors in $\mathbb{V}$ is a basis for $\mathbb{V}$.

(4) ——— p vectors in $\mathbb{V}$ that spans $\mathbb{V}$ is a basis for $\mathbb{V}$.

<u>Proof</u>: (1) Thm 1.

(2) If $S = \{v_1, \dots v_s\}$ spans with $s < p$ then $\# B = p$ would imply B is l.d by Theorem 2. This can't happen!

(3) If they didn't span, we could add a vector to it and remain l.i, contradicting (1).

(4) If they weren't l.i, we could remove one or more vectors and still span $\mathbb{V}$, Then $\dim \mathbb{V} < p$, a contradiction to our original assumption.

<u>§2 Application</u>: The rank of a matrix

$A = m \times n$ matrix $\rightsquigarrow$ $Null(A)$ subspace of $\mathbb{R}^n$ $\dim =:$ nullity of A

$\rightsquigarrow$ $Range(A)$ ——— $\mathbb{R}^m$ $\dim =: rank(A)$

<u>Q</u>: How to compute nullity & rank?

(1) <u>For Nullity</u>: Find $A \underset{\text{row equiv}}{\sim} A'$ REF nullity = # $\underset{\text{variables}}{\text{independent}}$ ($= n - rank(A')$.)

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 1 & 5 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ REF $\rightsquigarrow$ 2 indep vars $\boxed{\text{nullity} = 2}$ (Basis Null(A) $= \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$)

(2) <u>For Rank</u>: Find $A^T \underset{\text{row equiv}}{\sim} A''$ REF rank = # non-zero rows of A''.

<u>Example</u>: $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ REF $\boxed{\text{rank} = 2}$

<u>Theorem 3</u>: $rank(A) = rank(A^T)$ So column & row space of A have the same dimension

Consequence: rank(A) $\overset{\text{OLD definition (for REF)}}{=}$ # dependent variables
nullity(A) = # independent —— } so $\boxed{n = \#\,\mathrm{col}(A) = \mathrm{rank}(A) + \mathrm{null}(A)}$

Theorem 4: A system $A\underline{x} = \underline{b}$ in $\mathbb{R}^n$ is consistent if and only if
$\mathrm{rank}(A) = \mathrm{rank}(A|b)$

Proof Consistency means $\underline{b}$ in $Sp(\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A)) = \mathrm{Range}(A)$.

$$\dim Sp(\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A), \underline{b}) = \dim \mathrm{Range}(A|b) = \mathrm{rank}(A|b)$$
$$\mathrel{\rotatebox{90}{$\geq$}}$$
$$\dim Sp(\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A)) = \dim \mathrm{Range}(A) = \mathrm{rank}(A)$$

- Equality holds if & only if $\mathrm{rank}(A) = \mathrm{ran}(A|b)$
- ———————————— $\underline{b}$ in $\mathrm{Range}(A)$.

Consequence: An $n \times n$ matrix A is nonsingular if and only if $\mathrm{rank}(A) = n$

Proof: Nonsingular $\equiv$ n columns are li $\equiv$ $\mathrm{rank}(A) = n$.

Proof of Thm 3: Write $A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ $\quad a_i = [a_{i1} \cdots a_{in}]$ row-vector with n entries.

- Step 1: Show $\mathrm{rank}(A) \leq \mathrm{rank}(A^T)$.
- Write $k = \mathrm{rank}(A^T)$ so $W = Sp(a_1^t \ldots a_m^t)$ has $\dim = k$ & W is a subspace of $\mathbb{R}^n$
- By Thm 2 (2): $m \geq k$ Fix a basis $\mathcal{B} = \{w_1, \ldots, w_k\}$ for W. $\quad w_i = \begin{bmatrix} w_{i1} \\ w_{in} \end{bmatrix}$ for all i

Write each a_i^t as a linear combination of $\mathcal{B}$:

$$\left\{\begin{array}{l} a_1^T = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} = \alpha_{11} \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{1n} \end{bmatrix} + \alpha_{12} \begin{bmatrix} w_{21} \\ w_{22} \\ \vdots \\ w_{2n} \end{bmatrix} + \cdots + \alpha_{1k} \begin{bmatrix} w_{k1} \\ \vdots \\ w_{kn} \end{bmatrix} \\ \vdots \\ a_m^T = \begin{bmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} = \alpha_{m1} \begin{bmatrix} w_{11} \\ \vdots \\ w_{1n} \end{bmatrix} + \alpha_{m2} \begin{bmatrix} w_{21} \\ \vdots \\ w_{2n} \end{bmatrix} + \cdots + \alpha_{mk} \begin{bmatrix} w_{k1} \\ \vdots \\ w_{kn} \end{bmatrix} \end{array}\right.$$

- Take j^{th} component of each equation:

$$\left\{\begin{array}{l} a_{1j} = \alpha_{11} \boxed{w_{1j}} + \alpha_{12} w_{2j} + \cdots + \alpha_{1k} \boxed{w_{kj}} \\ \vdots \\ a_{mj} = \alpha_{m1} \boxed{w_{1j}} + \alpha_{m2} w_{2j} + \cdots + \alpha_{mk} \boxed{w_{kj}} \end{array}\right.$$

$\rightsquigarrow$ $\mathrm{col}_j(A)$ on (LHS) get $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = col_j(A) = w_{1j}\begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{bmatrix} + \cdots + w_{kj}\begin{bmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{mk} \end{bmatrix}$ for each j

Conclude: Range(A) is a subset of $Sp\left(\begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{mk} \end{bmatrix}\right) = \mathbb{W}$.

So $\text{Rank}(A) = \dim \text{Range}(A) \leq \dim \mathbb{W} \leq k = \text{Rank}(A^T)$.

Step 2: Use $(A^T)^T = A$ & Step 1 to get $\text{Rank}(A^T) \leq \text{Rank}(\underbrace{(A^T)^T}_{=A})$

Combine Step 1 & 2 to get $\text{rank}(A) \leq \text{rank}(A^T) \leq \text{rank}(A)$, so $\text{rank}(A) = \text{rank}(A^T)$. □

In particular: We get $\text{rank}(A') = \#$ dep vars for A' in REF

same row space → $\text{rank}(A) = \text{rank}(A')$ if $A \sim A'$ row-equiv.

· $\text{null}(A) = \text{null}(A')$ if $A \sim A'$

$\Big\}$ $\text{rank}\,A + \text{null}\,A = n$.

Consequence: METHOD 2 for finding a basis for $\mathbb{V} = Sp(v_1, \dots, v_r)$ subspace of $\mathbb{R}^n$ is correct.

Why? $A = [v_1, \dots v_r] \sim A'$ and $\text{rank}(A') = \text{rank}(A)$

$\text{rank}(A') = r - \text{null}(A')$, $\text{rank}(A) = r - \text{null}(A)$

$A' = \begin{bmatrix} \star \ \star \ \cdots \\ 0 \quad \star \ \star \ \cdots \\ \quad\quad \star \ \star \ \star \ \times \end{bmatrix}$ $k = \dim \underbrace{\text{Range}(A)}_{= \mathbb{V}}$

$i_1 \quad i_2 \quad i_3 \cdots i_k$
dependent variable

Then $\tilde{A} = [v_{i_1}\ v_{i_2}\ v_{i_3} \cdots v_{i_k}] \sim \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \\ 0 & 0 & 0 & 1 \\ \hline 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix}$ → columns are l.i. so $\dim \text{Range}\,\tilde{A} = K$

same row operations fitting $A \sim A'$. (K cols)

Conclude: $Sp\{v_{i_1} \cdots v_{i_k}\}$ has dim k, same as $\mathbb{V}$ so by Thm 2 we know they span $\mathbb{V}$ & so they are l.i.

$B = \{v_{i_1}, \dots, v_{i_k}\}$ is a basis, as the method predicted.