Lecture XXI: §§5.1-5.2 Vector Spaces

§1. Motivation:

So far: constructed $\mathbb{R}^n$ & vector subspaces of $\mathbb{R}^n$:

(1) $W = \mathbb{R}^n$

(2) $W = \{\mathbb{0}\}$

(3) $W = Sp(\vec{v}_1, \dots, \vec{v}_r)$

(4) $W = \mathcal{N}(\underset{m\times n}{A}) = \left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m\right\}$ Nullity Space

(5) $W = \text{Range}(A) = \text{Column Space}(A) = \underset{\text{subspace of } \mathbb{R}^m}{Sp(a_1, \dots, a_n)}$ $\quad \underset{m\times n}{A = [a_1 \cdots a_n]}$

(6) $W = \text{Row Space}(A) = \text{Column Space}(A^T)$ subspace of $\mathbb{R}^n$.

Common features:
- $\mathbb{0}$ in W
- We can add two vectors of W (using $+$ in $\mathbb{R}^n$) & remain in W
- —— scalar multiply a vector in W with any scalar (using the usual scalar product operation in $\mathbb{R}^n$) & remain in W.
- $+$ & scalar mult. have nice algebraic properties (inherited from $\mathbb{R}^n$)

We see these properties in other settings:

Example 1: A set of solutions to the differential equation $y''_{(x)} - y_{(x)} = 0$

- **Solution 1**: $y(x) = e^x$
- **Solution 2**: $y(x) = -e^{-x}$
- Any linear combination of these $y(x) = c_1 e^x + c_2(-e^{-x})$ is also a solution.

$$y'(x) = c_1 e^x + c_2 e^{-x}$$
$$y''(x) = c_1 e^x + c_2(-e^{-x}) = y(x) \quad \checkmark$$

- **Claim**: These 2 solutions are "linearly independent":

$$0 = c_1 e^x + c_2(-e^{-x}) \quad \text{as functions}$$

We want to conclude that $c_1 = c_2 = 0$.

We do so by evaluating at suitable values of x:

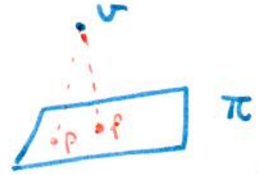
- **At $x=0$**: $0 = c_1 e^0 + c_2(-e^{-0}) = c_1 - c_2$, so $c_2 = c_1$

- At $x=1$ $\quad 0 = c_1 e + c_1(-e^{-1}) = c_1\underbrace{\left(e - \frac{1}{e}\right)}_{\neq 0}$ so $c_1 = 0$

Conclude: $c_1 = c_2 = 0$.

- Why do we care? We can show $\{ y \mid y'' + y = 0 \} = Sp(e^x, -e^{-x})$

Example 2: Fix



π plane through $(0,0,0)$ in $\mathbb{R}^3$

Q: What is the closest point p to v in the plane π?

A: Need to minimize $d_{(v,p)} = \|v - p\|$ for p in π
$$\|v-p\| = \sqrt{\langle v-p, v-p\rangle}$$

Answer $\overline{vp} \perp \pi$ so it must be proportional to $\vec{n}$ = normal to π.

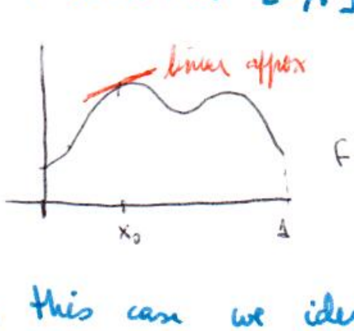
Similarly, fix the space V of all continuous functions over $[0,1]$

We have $P_2 = \{ a + bx + cx^2 ; a, b, c \text{ in } \mathbb{R}\}$ (polynomials of degree ≤ 2)

- Want to find $P = a + bx + cx^2$ minimizing $\|f - P\| = \sqrt{\langle f-P, f-P\rangle}$, ie distance from f to P_2.

We Take $\langle g, h\rangle = \int_0^1 g_{(x)} h_{(x)}\, dx$ (inner product for cont functions on $[0,1]$)

This extends the notion of linear approximation for a ~~continuous~~ continuous & differentiable function near a point x_0 (use tangent line / tangent plane to the graph of the function at the point $(x_0, f(x_0))$) to a quadratic approximation over the interval $[0,1]$



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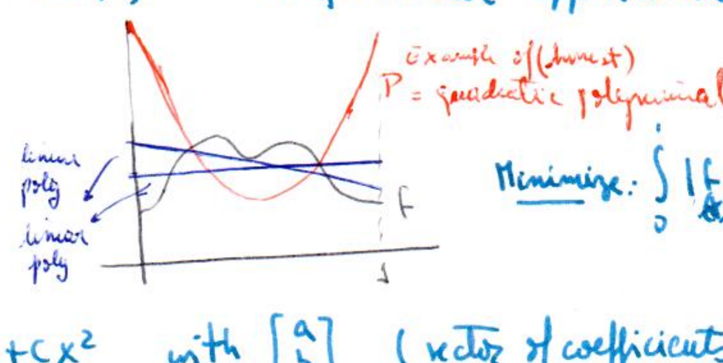

Minimize: $\int_0^1 |f_{(x)} - P_{(x)}|^2\, dx$

In this case we identify $a + bx + cx^2$ with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (vector of coefficients),

but we don't have the ~~usual~~ distance from $\mathbb{R}^3$ (usual dot product). Declare $\left\langle \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \right\rangle = \langle a + bx + cx^2, a' + b'x + c'x^2 \rangle$

§2. Vector Spaces:

A general vector space consists of a set of elements (called vectors) $\mathbb{V}$ & a set of scalars S ($\mathbb{R}$ or $\mathbb{C}$) with 2 operations:

(1) Addition: $+ \quad \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$(u, v) \longmapsto u+v$$

(2) Scalar Multiplication For $\mathbb{R}$: $\cdot : \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$(\alpha, v) \longmapsto \alpha v$$

Want $+$ & $\cdot$ to have nice algebraic properties.

Examples: • $\mathbb{R}^n$ & all subspaces discussed on page 1

• $\mathbb{R}^{n \times m}$ = matrices of size $n \times m$ with usual $+$ & scalar mult.

• Solutions to homogeneous differential equations (e.g. $y'' + y = 0$)

Formal definition: A set of elements $\mathbb{V}$ is a vector space over $\mathbb{R}$ if we can define addition & scalar multiplication on $\mathbb{V}$ & the following properties hold for any u, v, w in $\mathbb{V}$ & scalars α, β in $\mathbb{R}$:

• Closure properties:

(C1) $u+v$ is a vector in $\mathbb{V}$ for u, v in $\mathbb{V}$

(C2) αu ———— for u in $\mathbb{V}$, α in $\mathbb{R}$

• Properties of Addition:

(A1) $u + v = v + u$ [Commutative]

(A2) $u + (v+w) = (u+v) + w$ [Associative]

(A3) There is a vector $\mathbb{0}$ in $\mathbb{V}$ such that

$$v + \mathbb{0} = \mathbb{0} + v = v \quad \text{for all } v \text{ in } \mathbb{V}$$

$\mathbb{0}$ = Neutral Element for $\mathbb{V}$

(A4) Given v in $\mathbb{V}$ there is w in $\mathbb{V}$ such that $v + w = w + v = \mathbb{0}$

w = Additive inverse for v

• Properties of Scalar Multiplication:

(M1) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$ [Commutative]

(M2) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$ [Distributive 1]

(M3) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$ [Distributive 2]

(M4) $1 \cdot v = v$ for all v

Example 1: $W = \{ \mathbb{0} \}$ with $\mathbb{0} + \mathbb{0} = \mathbb{0}$, $\alpha \cdot \mathbb{0} = \mathbb{0}$ satisfies all 10 properties

Example 2: $n \times n$ matrices, $\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdot & & \cdots & 0 \end{bmatrix}$ with usual + & $\cdot$.

Example 3: $V = \mathbb{R}^2$ with funky addition $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix}$

& usual scalar multiplication

- (C1), (C2) holds
- (A1) holds
- (A2) $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 + w_1 - 1 \\ v_2 + w_2 - 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$

  $\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$

  $=$

- (A3) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 - 1 \\ v_2 + u_2 - 1 \end{bmatrix} \rightsquigarrow \begin{cases} v_1 = v_1 + u_1 - 1 \\ v_2 = v_2 + u_2 - 1 \end{cases} \rightsquigarrow u_1 = u_2 = 1$

  Neutral Element $\mathbb{0}_{new} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- (A4) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 - 1 \\ v_2 + w_2 - 1 \end{bmatrix} \rightsquigarrow \begin{matrix} v_1 + w_1 - 1 = 1 \\ v_2 + w_2 - 1 = 1 \end{matrix}$ so $\begin{matrix} w_1 = 2 - v_1 \\ w_2 = 2 - v_2 \end{matrix}$

  So inverse for $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 - v_1 \\ 2 - v_2 \end{bmatrix}$

- (M1) holds
- (M2) $\alpha \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \alpha \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - \alpha \\ \alpha u_2 + \alpha v_2 - \alpha \end{bmatrix}$

  $\alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - 1 \\ \alpha u_2 + \alpha v_2 - 1 \end{bmatrix}$

  To get = we must have $\alpha = 1$ But scalars are arbitrary!

  Conclude: (M2) fails.

  (M3) similarly fails

  (M4) holds

Conclude: $(\mathbb{R}^2, \oplus, \cdot)$ is NOT a vector space.

Example 4: $V = \{ 2 \times 2$ singular matrices $\}$ is NOT a vector space

(M1) — (M4), (A1), (A2) hold

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular & its $\mathbb{0}$, so (A3) holds

(A4) A singular, Additive inverse $= -A$ is also singular.

(C2) holds A singular, then αA also singular

$\{col_1 A, col_2 A\}$ l.d. $\{\alpha\, col_1 A, \alpha\, col_2 A\}$ also l.d.

(C1) fails: $\underset{\text{sing}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} + \underset{\text{sing}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{\text{non-sing}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ so W is not closed under +.

Example 5: $P_2 = \{ a + bx + cx^2 : a, b, c \text{ in } \mathbb{R} \}$ = polynomials of degree ≤ 2

$+ : (a + bx + cx^2) + (a' + b'x + c'x^2) = (a+a') + (b+b')x + (c+c')x^2$

$\cdot \quad \alpha(a + bx + cx^2) = (\alpha a) + (\alpha b)x + (\alpha c)x^2$

. (C1), (C2) hold, (A1), (A2), (M1) — (M4) hold

(A3) $\mathbb{O} = 0 + 0 \cdot x + 0 \cdot x^2$ neutral element

(A4) $-P = (-1)P$ additive inverse

$P_2 = Sp(1, x, x^2)$