Lecture XXII: §5.3 Vector Subspaces

Last time: • Defined abstract vector spaces $(\mathbb{V}, +, \cdot) \to$ Closure Properties

$\searrow$ Additive Props } "easy

$\searrow$ Scalar Mult " } algebra"

& $\mathbb{0}$, $-v$

• Many examples & non-examples.

- Matrices $= M_{m \times n}$
- polynomials of bounded degree (fixed) $\mathcal{P}_n = \{ a_0 + a_1 x + \cdots + a_n x^n ; a_0, \ldots, a_n \text{ in } \mathbb{R} \}$ $\longleftrightarrow$ $\begin{bmatrix} a_0 \\ a_n \end{bmatrix}$ in $\mathbb{R}^{n+1}$

§1 More useful properties

① **Cancellation Laws:**

(1) If $u + v = u + w$, then $v = w$

Why? Take $u' = -u$ (additive inverse) Then $\underbrace{u' + (u}_{=\mathbb{0}} + v) = v$

$\|$

$u' + (u + w) = w$

(2) If $v + u = w + u$ then $v + w$

Why? (1) & + is commutative. (A1)

② **Theorem:** (1) The zero vector $\mathbb{0}$ (= Neutral element for + in $\mathbb{V}$) is unique

(2) The additive inverse $\overset{-v}{}$ for v is unique & $-v = (-1) \cdot v$

(3) $0 \cdot v = \mathbb{0}$ for all v in $\mathbb{V}$

(4) $\alpha \cdot \mathbb{0} = \mathbb{0}$ for all scalars α in $\mathbb{R}$

(5) If $\alpha \cdot v = \mathbb{0}$, then either $\alpha = 0$ or $v = \mathbb{0}$.

Proof: (1) Say $\mathbb{0}, \mathbb{0}'$ are neutral elements for +.

Then: $\mathbb{0} \underset{\substack{\downarrow \\ \mathbb{0}' \text{ neutral elem} \\ v = \mathbb{0}}}{=} \mathbb{0} + \mathbb{0}' \underset{\substack{\downarrow \\ \mathbb{0} \text{ neutral elem} \\ v = \mathbb{0}'}}{=} \mathbb{0}'$

(2) • Additive inverses are unique: If w, w' satisfy $v + w = v + w' = \mathbb{0}$

then, $w = w + \mathbb{0} = (w + v) + w' = \mathbb{0} + w' = w'$

So we write the inverse as $-v$.

• $(-1)\, v$ satisfies $v + (-1) \cdot v = 1 \cdot v + (-1) \cdot v = (1 - 1) \cdot v = 0 \cdot v \underset{\substack{\uparrow \\ (3)}}{=} \mathbb{0}$.

(3) Write $w = 0 \cdot v$

Then $w = 0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v = w + w$

Write w' for the inverse for w. Then, $\mathbb{0} = w + w' = w + (w + w') = w + \mathbb{0} = w$.

(4) Again, write $\omega = \alpha\cdot\mathbb{0}$

Then $\omega = \alpha\cdot(\mathbb{0}) = \alpha(\mathbb{0}+\mathbb{0}) = \alpha\cdot\mathbb{0} + \alpha\cdot\mathbb{0} = \omega + \omega$

Again write $\omega' = -\omega$, then $\mathbb{0} = \omega + (-\omega) = \omega + (\underbrace{\omega + (-\omega)}_{=\mathbb{0}}) = \omega$.

(5) If $\alpha \neq 0$, then $\mathbb{0} = \frac{1}{\alpha}(\underbrace{\alpha\cdot v}_{=\mathbb{0}}) = \frac{\alpha}{\alpha}\cdot v = 1\cdot v = v$. So, $v = \mathbb{0}$.

Conclusion: either $\alpha = 0$ or $v = \mathbb{0}$.

§2. Subspaces:

Next step: Fix a vector space $\mathbb{V}$ (eg $\mathbb{R}^n$) & want to find conditions for a subset $\mathbb{W}$ to be a vector space with inherited + & ·.

- Need:
  - + to be an operation in $\mathbb{W}$ (C1)
  - · ———————— $\mathbb{W}$ (C2)
  - a neutral element in $\mathbb{W}$. It must be $\mathbb{0} = 0\cdot w$ for any w in $\mathbb{W}$. (A3)
  - additive inverses in $\mathbb{W}$: $-w = (-1)w$ follows from (C2).

As with $\mathbb{R}^n$ & subspaces of $\mathbb{R}^n$, we only need to check 3 things:

Theorem: Fix $\mathbb{W}$ a subset of $\mathbb{V}$, and $\mathbb{V}$ a vector space. Then $\mathbb{W}$ with inherited + & · is a subspace of $\mathbb{V}$ if and only if:

(S1) $\mathbb{0}$ from $\mathbb{V}$ lies in $\mathbb{W}$

(S2) given u, v in $\mathbb{W}$, we have $u+v$ in $\mathbb{W}$.

(S2) ——— u in $\mathbb{W}$ & α in $\mathbb{R}$ we have $\alpha\cdot u$ in $\mathbb{W}$.

Here: subspace means vector space on itself with inherited + & ·.

Examples:

① $C_{[a,b]}$: $\{f : [a,b] \to \mathbb{R}$ continuous$\}$ with pointwise + & ·

- $f + g : [a,b] \to \mathbb{R}$ $\quad (f+g)_{(x)} = f_{(x)} + g_{(x)}$ for all x in $[a,b]$
- $\alpha f : [a,b] \to \mathbb{R}$ $\quad (\alpha f)_{(x)} = \alpha f_{(x)}$ ————————
- $\mathbb{0} : [a,b] \to \mathbb{R}$ constant zero function
- All 10 properties are satisfied.
- $\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2\} = \mathrm{Sp}\{1, x, x^2\}$ is a subspace of $C[a,b]$. Same for $\mathcal{P}_n = \{a_0 + a_1 x + \dots + a_n x^n\}$

② $M_{2\times 3} = \{ 2\times 3 \text{ matrices} \}$ is a vector space $\left(\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$

- $W_2 = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\}$ is a subspace.

(S2) $\underbrace{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix}}_{\text{in } W} + \underbrace{\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \end{pmatrix}}_{\text{in } W} = \begin{pmatrix} a_{11}+b_{11} & 0 & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & 0 \end{pmatrix}$ in W

(S3) $\alpha \underbrace{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix}}_{\text{in } W} = \begin{pmatrix} \alpha a_{11} & 0 & \alpha a_{13} \\ 0 & \alpha a_{22} & 0 \end{pmatrix}$ in W

(S1) $\mathbb{0}$ in W for $a_{11} = a_{13} = a_{22} = 0$.

③ $V = \mathcal{P}_2$, $W_3 = \{ P_{(x)} \text{ in } \mathcal{P}_2 : P'_{(0)} = 0 \}$ is a subspace of $\mathcal{P}_2$

Why? $P_{(x)} = a + bx + cx^2 \rightsquigarrow P'_{(x)} = b + 2cx$ $0 = P'_{(0)} = b$

Conclude: $W = \{ a + cx^2 \} = Sp(1, x^2)$

④ $W_4 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{11}a_{22} - a_{21}a_{12} = 0 \right\}$ is **NOT** a subspace of $M_{2\times 3}$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ in W, but $A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **not** in W

$(1\cdot 1 - 0 \cdot 0 = 1 \neq 0)$

⑤ $W_5 = \{ f \text{ in } C[a,b] : \int_a^b f_{(x)}\, dx = 0 \}$ is a subspace of $C[a,b]$

because $\int_a^b$ has linear behavior.

⑥ $W_6 = \{ f \text{ in } C[a,b] : f'_{(1)} = 0 \}$ is a subspace of $C[a,b]$

(S1) $f_{(x)} \equiv \mathbb{0}$ then $f'_{(x)} = \mathbb{0}'_{(x)} = 0$ so $f'_{(1)} = 0$.

(S2) $(f+g)'_{(x)} = f'_{(x)} + g'_{(x)}$ so $(f+g)'_{(1)} = 0 + 0 = 0$ for f, g in W

(S3) $(\alpha f)'_{(x)} = \alpha f'_{(x)}$ so $(\alpha f)'_{(1)} = \alpha f'_{(1)} = \alpha \cdot 0 = 0$ for f in W, α in $\mathbb{R}$.

§3. Spanning Sets:

Same definition as for $\mathbb{R}^n$.

Def: A vector v in V is a linear combination of $v_1, \dots v_r$ vectors in V if

$v = \alpha_1 v_1 + \dots + \alpha_r v_r$ for scalars $\alpha_1, \dots \alpha_r$.

Write: $Sp(v_1, \dots v_r)$ for the set of all linear combinations of $v_1, \dots, v_r$.

Ex: $\mathcal{P}_2 = Sp(1, x, x^2)$.

Def $\{v_1, \dots v_r\}$ **spans** $\mathbb{V}$ if $\mathbb{V} = Sp(v_1, \cdots v_r)$

Warning: not all vector spaces admit (finite) spanning sets

Example: $C[0,1]$ has no spanning set (we'll need $1, x, x^2, x^3, \dots$ and many more!)

Examples from earlier:

- $$\mathbb{M}_{2\times 3} = Sp\left(\underset{\overset{\shortparallel}{E_{11}}}{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}}, \underset{\overset{\shortparallel}{E_{12}}}{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}}, \underset{\overset{\shortparallel}{E_{13}}}{\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}}, \underset{\overset{\shortparallel}{E_{21}}}{\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}}, \underset{\overset{\shortparallel}{E_{22}}}{\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}}, \underset{\overset{\shortparallel}{E_{23}}}{\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}} \right\}$$

  E_{ij} = matrix A in $\mathbb{M}_{m\times n}$ with 1 in (i,j) entry & 0's everywhere else.

  They play the role that $\underset{(e_1, \dots, e_N)}{\underline{\text{basic unit vectors}}}$ did for $\mathbb{R}^N$

- $\mathbb{W} = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{bmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\} = Sp(E_{11}, E_{13}, E_{22})$

- $C[a,b]$ has no spanning set, neither do W_5, W_6

Theorem 2: If $\mathbb{V}$ is a vector space & $\{v_1, \cdots v_r\}$ are vectors in $\mathbb{V}$, then $\mathbb{W} = Sp(v_1 \cdots v_r)$ is a subspace of $\mathbb{V}$.

Proof: (S1)

$$\begin{array}{rl} & u = \alpha_1 v_1 + \cdots + \alpha_r v_r \qquad (\text{in } Sp(v_1, \cdots v_r)) \\ + & \underline{v = \beta_1 v_1 + \cdots + \beta_r v_r} \qquad (\rule{3em}{0.4pt}) \\ & u+v = (\alpha_1+\beta_1)v_1 + \cdots + (\alpha_r + \beta_r)v_r \text{, so in } Sp(v_1, \cdots v_r). \end{array}$$

(S2) $\alpha \cdot v = (\alpha\beta_1)v_1 + \cdots + (\alpha\beta_r)v_r$ so also in ———.

(S3) $\mathbb{0} \overset{?}{=} \alpha_1 v_1 + \cdots + \alpha_r v_r$ YES, Take $\alpha_1 = \cdots = \alpha_r = 0$.

Since (S1), (S2) & (S3) hold, $\mathbb{W}$ is a subspace of $\mathbb{V}$.