# Lecture XXIII: §5.4 Linear Independence & Bases

§0. Review:

Last Time: • Defined subspace of an abstract vector space $\mathbb{V}$

(S1) If $u, v$ in $\mathbb{W}$, then $u+v$ also in $\mathbb{W}$

(S2) If $u$ in $\mathbb{W}$ & $\alpha$ in $\mathbb{R}$ then $\alpha u$ also in $\mathbb{W}$

(S3) $\mathbb{O}$ in $\mathbb{W}$

Main example: $Sp(v_1, \ldots v_r) = \{\alpha_1 v_1 + \cdots + \alpha_r v_r : \alpha_1, \ldots, \alpha_r \text{ in } \mathbb{R}\}$

• $C_{[0,1]}$ has no finite spanning sets.

Examples ① $\{1, x, x^2\}$ spans $\mathcal{P}_2 = \{a + bx + cx^2 : a, b, c \text{ in } \mathbb{R}\}$

② $\mathbb{W} = \{A \text{ in } M_{3\times 3} : A^T = A\}$  We build a spanning set

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\|$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

This gives: $a_{12} = a_{21}$
$a_{13} = a_{31}$
$a_{23} = a_{32}$

Take $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}$ out

$$\text{So } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{11}} + a_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{12}+E_{21}} + a_{13} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=E_{13}+E_{31}}$$

$$+ a_{22} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{22}} + a_{23} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{=E_{23}+E_{32}} + a_{33} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=E_{33}}$$

Conclude: $\mathbb{W} = Sp(E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33})$

## §1. Linear independence:

Def: Fix a vector space $\mathbb{V}$ & $v_1, \ldots, v_r$ vectors in $\mathbb{V}$. We write:

$$(*) \qquad \mathbb{O} = \boxed{\alpha_1} v_1 + \cdots + \boxed{\alpha_r} v_r$$

We know $\alpha_1 = \cdots = \alpha_r = 0$ is a solution.

• The set $\{v_1, \ldots, v_r\}$ is <u>linearly independent</u> if the only solution to $(*)$ is the trivial one, $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

• Otherwise, we say $\{v_1, \ldots, v_r\}$ is <u>linearly dependent</u>.

Obs: This is the same definition as for $\mathbb{R}^n$. We can decide l.i vs l.d with the same methods!

METHOD 2: Subsets of $M_{m\times n}$

① $\{E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33}\}$ is l.i

Why? Write $\mathbb{0} = a_{11}E_{11} + a_{12}(E_{12}+E_{21}) + a_{13}(E_{13}+E_{31}) + a_{22}E_{22} + a_{23}(E_{23}+E_{32}) + a_{33}E_{33}$

$$\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}$$ undo the "splitting" done in page 2.

Conclude all entries agree, so $a_{11}=a_{12}=a_{13}=a_{22}=a_{23}=a_{33}=0$ is the unique solution to the equation (*).

② $\left\{ v_1 = \begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}, v_2 = \begin{bmatrix}1&0&2\\0&1&0\end{bmatrix}, v_3 = \begin{bmatrix}3&0&5\\0&3&0\end{bmatrix}, v_4 = \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix} \right\}$ is l.d.

$$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix} = \mathbb{0} = a\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix} + b\begin{bmatrix}1&0&2\\0&1&0\end{bmatrix} + c\begin{bmatrix}3&0&5\\0&3&0\end{bmatrix} + d\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix} a+b+3c & 0 & a+2b+5c+d\\ 0 & a+b+3c & 0\end{bmatrix}$$

We get $\begin{cases} a+b+3c=0\\ a+2b+5c+d=0\\ a+b+3c=0\end{cases} \equiv \begin{cases} a+b+3c=0\\ a+2b+5c+d=0\end{cases}$

2 equations, 4 unknowns, homogeneous $\rightsquigarrow \geq 2$ independent variables

So we have infinitely many solutions, & so $\{v_1,v_2,v_3,v_4\}$ is l.d.

Explicit solutions:

$$\begin{bmatrix}1&1&3&0\\1&2&5&1\end{bmatrix} \xrightarrow[R_2\to R_2-R_1]{} \begin{bmatrix}1&1&3&0\\0&1&2&1\end{bmatrix} \xrightarrow[R_1\to R_1-R_2]{} \begin{bmatrix}1&0&1&-1\\0&1&2&1\end{bmatrix} \quad \text{REF}$$

so $\begin{cases} a = -c+d\\ b=-2c-d\end{cases}$

Solns: $$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}-c+d\\-2c-d\\c\\d\end{bmatrix} = c\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix} + d\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}$$

We get $-1v_1 - 2v_2 + v_3 = \mathbb{0}$, $v_1 - v_2 + v_4 = \mathbb{0}$ — 2 "generating" linear relations

$\rightsquigarrow Sp(v_1,v_2,v_3,v_4) = Sp(v_1,v_2)$ because $v_3 = v_1+2v_2$, $v_4 = -v_1+v_2$

**METHOD 2**: For subspaces of functions

① $\{1, x, x^2\}$ is l.i in $\mathbb{P}_2$: $\quad 0 = a + bx + cx^2$

We evaluate at convenient values for $x$:

$$\begin{aligned} x=0 &: \quad 0 = a \\ x=1 &: \quad 0 = a+b+c \\ x=-1 &: \quad 0 = a-b+c \end{aligned} \qquad \left.\begin{aligned} &= b+c \\ &= -b+c \end{aligned}\right\} \qquad \begin{aligned} &a=0 \\ &b=c=0 \text{ is only solution} \end{aligned}$$

② $\{1, (x+1)^2, (x-1)^2, x^2\}$ is l.d.

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$$

Option 1: Open up squares, regroup scalars & use $\{1, x, x^2\}$ is l i

$$0 = a + b(x^2+2x+1) + c(x^2-2x+1) + dx^2$$

$$0 = (a+b+c) + (2b-2c)\,x + (b+c+d)\,x^2$$

So $\begin{cases} a+b+c=0 & \rightsquigarrow a+2b=0 \rightsquigarrow a=-2b \\ 2b-2c=0 & \rightsquigarrow b=c \\ b+c+d=0 & \rightsquigarrow 2b+d=0 \rightsquigarrow d=-2b \end{cases}$

So $0 = b\left(-2 + (x+1)^2 + (x-1)^2 - 2x^2\right)$ for all $b$

$\boxed{0 = -2 + (x+1)^2 + (x-1)^2 - 2x^2}$. nontrivial relation.

Option 2: Evaluate at convenient values of $x$: (pick values where some vectors vanish)

$$\begin{aligned} \text{At } x=0 &: \quad a+b+c=0 \\ \text{At } x=1 &: \quad a+4b+d=0 \\ \text{At } x=-1 &: \quad a+4c+d=0 \\ \text{At } x=2 &: \quad a+9b+c+4d=0 \end{aligned}$$

solve the system for $a, b, c, d$

$$\left[\begin{array}{cccc|c} 1&1&1&0&0\\ 1&4&0&1&0\\ 1&0&4&1&0\\ 1&9&1&4&0 \end{array}\right] \xrightarrow[\substack{R_2\to R_2-R_1\\ R_3\to R_3-R_1\\ R_4\to R_4-R_1}]{} \left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&3&-1&1&0\\ 0&-1&3&1&0\\ 0&8&0&4&0 \end{array}\right] \xrightarrow[\substack{R_3\to \frac{R_3}{2}\\ R_4\to \frac{R_4}{4}\\ R_3\leftrightarrow R_2}]{} \left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-3&-1&0\\ 0&3&-1&1&0\\ 0&2&0&1&0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3\to R_3-3R_2\\ R_4\to R_4-2R_2}]{} \left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-3&-1&0\\ 0&0&8&4&0\\ 0&0&6&3&0 \end{array}\right] \xrightarrow[\substack{R_3\to \frac{R_3}{8}\\ R_4\to R_4-6R_3}]{} \left[\begin{array}{cccc|c} 1&1&1&0&0\\ 0&1&-3&-1&0\\ 0&0&1&\frac12&0\\ 0&0&0&0&0 \end{array}\right] \xrightarrow[\substack{R_2\to R_2+3R_3\\ R_1\to R_1-R_3\\ R_1\to R_1-R_2}]{} \left[\begin{array}{cccc} 1&0&0&-1\\ 0&1&0&\frac12\\ 0&0&1&\frac12\\ 0&0&0&0 \end{array}\right]$$

$\rightsquigarrow \boxed{a=d,\ b=c=-\tfrac12 d} \overset{d=1}{\rightsquigarrow} \boxed{0 = \left(1 - \tfrac12(1+x)^2 - \tfrac12(x-1)^2 + x^2\right)}$ Relation

**Option 3**: Take derivatives to get relations:

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx^2 \qquad (1)$$

$$\frac{d}{dx} \quad 0 = 2b(x+1) + 2c(x-1) + 2dx = 2(b+c+d)x + 2(b-c) \qquad (2)$$

$$\frac{d}{dx} \quad 0 = 2b + 2c + 2d = 2(b+c+d) \qquad (3)$$

Take (2): get
$b+c+d = 0 \rightsquigarrow d = -2b$
$b - c = 0 \rightsquigarrow b = c$

Back to (1):
$$0 = a + b((x+1)^2 + (x-1)^2) - 2bx^2$$
$$0 = a + b((x+1)^2 + (x-1)^2 - 2x^2)$$
$$0 = a + b(2) \qquad \text{so } a = -2b.$$

**Conclude**: $a = d = -2b$, $c = b$

**Relation** (for $b=1$): $\boxed{0 = -2 + (x+1)^2 + (x-1)^2 - 2x^2}$ (double of relation obtained using Option 2)

## §3. Bases for abstract vector spaces:

**Def**: Let $\mathbb{V}$ be a vector spaces. A set $B = \{v_1, \ldots, v_r\}$ is a **basis** for $\mathbb{V}$ if
(1) $B$ is a spanning set for $\mathbb{V}$
(2) $B$ is linearly independent (equivalently: $B$ is a **minimal** spanning set)

**Examples**:
- $M_{2\times 3}$ has basis $\{E_{ij} : \begin{smallmatrix} 1 \le i \le 2 \\ 1 \le j \le 3\end{smallmatrix}\}$ (6 elements)
- $M_{m\times n}$ has basis $\{E_{ij} \ \begin{smallmatrix} 1 \le i \le m \\ 1 \le j \le n\end{smallmatrix}\}$ ($m \cdot n$ elements)
- $\mathcal{P}_d = \{a_0 + a_1 x + \cdots + a_d x^d\}$, $a_0, a_1, \ldots, a_d$ in $\mathbb{R}$, has basis $\{1, x, x^2, \ldots, x^d\}$ ($d+1$ elements)

**Upshot**: Basis of a vector space $\mathbb{V}$ with a finite spanning set all have the same number of element. We define this number as the **dimension** of $\mathbb{V}$

**ALGORITHM**: Start from a spanning set $S = \{v_1, \ldots, v_r\}$ for $\mathbb{V}$
(1) If $S$ is L.I, then $S$ is a basis
(2) ——— L.D, use a relation to write some $v_i$ as a linear comb. of the others. Set $S' = \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\}$ $S'$ If LI, done; If LD, ....

<u>Example</u>: $S = \{1, (x+1)^2, (x-1)^2, x\}$ & $W = \mathcal{P}_2$

- S spans $\mathcal{P}_2$:

$$\begin{cases} 1 \in Sp(S) \\ x \in Sp(S) \\ x^2 = (x+1)^2 - 2x - 1 \quad \in Sp(S) \end{cases}$$

since a basis for $\mathcal{P}_2$ can be generated from S, then S is a spanning set.

- S is L.D.:

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx$$

$$\frac{d}{dx} 0 = 2b(x+1) + 2c(x-1) + d = 2(b+c)x + 2(b-c) + d = 2(+2b) + d$$

$$= +4b + d$$

$$\frac{d}{dx} 0 = 2(b+c) \rightsquigarrow b + c = 0 \rightsquigarrow -b = +c$$

so $\boxed{d = -4b}$

Replace in original equation:

$$0 = a + b(x+1)^2 - b(x-1)^2 - 4bx$$

$$0 = a + b(\underbrace{(x+1)^2 - (x-1)^2 - 4x}_{=0}) \quad \text{so } a = 0$$

<u>Conclude</u> there is a unique relation (set $b=1$)

$$0 = (x+1)^2 - (x-1)^2 - 4x$$

So $(x+1)^2 = (x-1)^2 + 4x \rightsquigarrow S' = \{1, (x-1)^2, x\}$ spans & l.i.

So $(x-1)^2 = -(x+1)^2 - 4x \rightsquigarrow S_2 = \{1, (x+1)^2, x\}$ is a basis

$x = \frac{1}{4}(x+1)^2 - \frac{1}{4}(x-1)^2 \rightsquigarrow S_3 = \{1, (x+1)^2, (x-1)^2\}$ is a basis

<u>Note</u>: I can remove any vector from S <u>except</u> 1 & get basis for $\mathcal{P}_2$